

THE SPECTRAL PROPERTIES OF THE STRONGLY COUPLED STURM HAMILTONIAN OF EVENTUALLY CONSTANT TYPE

YAN-HUI QU

ABSTRACT. We study the spectral properties of the Sturm Hamiltonian of eventually constant type, which includes the Fibonacci Hamiltonian. Let s be the Hausdorff dimension of the spectrum. For $V > 20$, we show that the restriction of the s -dimensional Hausdorff measure to the spectrum is a Gibbs type measure; the density of states measure is a Markov measure. Based on the fine structures of these measures, we show that both measures are exact dimensional; we obtain exact asymptotic behaviors for the optimal Hölder exponent and the Hausdorff dimension of the density of states measure and for the Hausdorff dimension of the spectrum. As a consequence, if the frequency is not silver number type, then for V big enough, we establish strict inequalities between these three spectral characteristics.

We achieve them by introducing an auxiliary symbolic dynamical system and applying the thermodynamical and multifractal formalisms of almost additive potentials.

Keywords: Sturm Hamiltonian; eventually constant type; Hausdorff dimension; density of states measure; optimal Hölder exponent; Gibbs type measure

1. INTRODUCTION

The Sturm Hamiltonian is a discrete Schrödinger operator

$$(H_{\alpha,V,\phi}\psi)_n := \psi_{n-1} + \psi_{n+1} + v_n\psi_n$$

on $\ell^2(\mathbb{Z})$, where the potential $(v_n)_{n \in \mathbb{Z}}$ is given by

$$(1.1) \quad v_n = V\chi_{[1-\{\alpha\},1)}(n\alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z},$$

where $\alpha > 0$ is irrational, and is called *frequency* ($\{\alpha\}$ is the fractional part of α), $V > 0$ is called *coupling*, $\phi \in [0, 1)$ is called *phase*. It is well-known that the spectrum of Sturm Hamiltonian is independent of ϕ , which we denote by $\Sigma_{\alpha,V}$ (see [4]).

Sturm Hamiltonian is firstly introduced by physicists to model the quasicrystal system, see [4] and the references therein for an excellent introduction about the physical background. From the mathematical point of view, one is interested in its spectral properties.

For a discrete Schrödinger operator, several spectral objects are very important, which are the spectrum, the spectral measure and the density of states measure of the related operator (see [8] for the exact definitions and related properties). Let us recall the definition of the density of states measure in our setting. By the spectral theorem, there are Borel probability measures $\mu_{\alpha,V,\phi}$ on \mathbb{R} such that

$$\langle \delta_0, g(H_{\alpha,V,\phi})\delta_0 \rangle = \int_{\mathbb{R}} g(x) d\mu_{\alpha,V,\phi}(x)$$

for all bounded measurable functions g , where δ_0 is the element in $\ell^2(\mathbb{Z})$ which takes value 1 at site 0 and 0 elsewhere. The *density of states measure* $\mathcal{N}_{\alpha,V}$ is given by the ϕ -average of these measures with respect to Lebesgue measure, that is,

$$\int_{\mathbb{T}} \langle \delta_0, g(H_{\alpha,V,\phi}) \delta_0 \rangle d\phi = \int_{\mathbb{R}} g(x) d\mathcal{N}_{\alpha,V}(x)$$

for all bounded measurable functions g . It is well-known that the density of states measure $\mathcal{N}_{\alpha,V}$ is continuous and supported on $\Sigma_{\alpha,V}$ (see for example [8]).

We will study the fractal properties of $\Sigma_{\alpha,V}$ and $\mathcal{N}_{\alpha,V}$ for a special class of α . See [17, 18] for various definitions of fractal dimensions of set and measure. Write $s_V(\alpha) := \dim_H \Sigma_{\alpha,V}$ and $d_V(\alpha) := \dim_H \mathcal{N}_{\alpha,V}$ for the Hausdorff dimensions of $\Sigma_{\alpha,V}$ and $\mathcal{N}_{\alpha,V}$, respectively; denote by $\gamma_V(\alpha)$ the optimal Hölder exponent of $\mathcal{N}_{\alpha,V}$ (see (1.6) for the definition). Before introducing our results, let us give a brief survey about the known results on Sturm Hamiltonian.

The most prominent model among the Sturm Hamiltonian is the Fibonacci Hamiltonian, for which the frequency is taken to be the golden number $\alpha_1 = (\sqrt{5}+1)/2$. This model was introduced by physicists to model the quasicrystal system, see [26, 35]. Sütö [39] showed that the spectrum has zero Lebesgue measure for all $V > 0$. Then it is natural to ask what is the fractal dimension of the spectrum. It follows implicitly from [6] and [38] that if $V \geq 16$, then $\dim_B \Sigma_{\alpha_1,V} = \dim_H \Sigma_{\alpha_1,V}$, where $\dim_B E$ denotes the Box-dimension of E (see also [9] for explicit statement). Raymond [37] first estimated the Hausdorff dimension, he showed that $\dim_H \Sigma_{\alpha_1,V} < 1$ for $V > 4$. Jitomirskaya and Last [24] showed that for any $V > 0$, the spectral measure of the operator has positive Hausdorff dimension, as a consequence $\dim_H \Sigma_{\alpha_1,V} > 0$. By using a dynamical method, Damanik et al. [9] got lower and upper bounds for the dimensions. Due to these bounds they further showed that

$$(1.2) \quad \lim_{V \rightarrow \infty} s_V(\alpha_1) \ln V = \ln(1 + \sqrt{2}).$$

By studying the hyperbolicity of Fibonacci trace map T_V restricting to some invariant surface S_V for small V , Damanik and Gorodetski [10] showed that $s_V(\alpha_1)$ is a C^∞ function of V on $(0, V_0)$ for some $V_0 > 0$. Motivated by totally different problem, Cantat [5] also studied the Fibonacci trace map (indeed more general), he showed that the T_V is hyperbolic for any $V > 0$. Combining with most recent works [36] and [14], now it is known that $s_V(\alpha_1)$, as a function of V , is analytic on $(0, \infty)$ and takes values in $(0, 1)$. In [11], Damanik and Gorodetski further showed that $\lim_{V \downarrow 0} s_V(\alpha_1) = 1$. In [12], Damanik and Gorodetski showed that $\mathcal{N}_{\alpha_1,V}$ is exact dimensional and $d_V(\alpha_1) < s_V(\alpha_1)$ for small V . In [13], Damanik and Gorodetski showed that $\gamma_V(\alpha_1) \rightarrow 1/2$ as $V \rightarrow 0$ and

$$(1.3) \quad \lim_{V \rightarrow \infty} \gamma_V(\alpha_1) \ln V = \frac{3}{2} \ln \alpha_1.$$

Now we turn to the general Sturm Hamiltonian case. Fix an irrational $\alpha > 0$ with continued fraction expansion $[a_0; a_1, a_2, \dots]$. Bellissard et al. [4] showed that $\Sigma_{\alpha,V}$ is a Cantor set of Lebesgue measure zero. Damanik, Killip and Lenz [15] showed that, if $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k a_i < \infty$, then $\dim_H \Sigma_{\alpha,V} > 0$. Based on the analysis of Raymond about the

structure of spectrum [37], the fractal dimensions of the spectrum of Sturm Hamiltonian were extensively studied in [29, 30, 27, 19, 28]. The current picture is the following. Write

$$K_*(\alpha) = \liminf_{k \rightarrow \infty} \left(\prod_{i=1}^k a_i \right)^{1/k} \quad \text{and} \quad K^*(\alpha) = \limsup_{k \rightarrow \infty} \left(\prod_{i=1}^k a_i \right)^{1/k}.$$

Fix $V \geq 24$. Then it is proven in [29, 28] that

$$\begin{cases} \dim_H \Sigma_{\alpha,V} \in (0,1) & \text{if } K_*(\alpha) < \infty \\ \dim_H \Sigma_{\alpha,V} = 1 & \text{if } K_*(\alpha) = \infty \end{cases} \quad \text{and} \quad \begin{cases} \overline{\dim}_B \Sigma_{\alpha,V} \in (0,1) & \text{if } K^*(\alpha) < \infty \\ \overline{\dim}_B \Sigma_{\alpha,V} = 1 & \text{if } K^*(\alpha) = \infty \end{cases}.$$

where $\overline{\dim}_B E$ denote the upper Box-dimension of E . Raymond [37], Liu and Wen [29] showed that the spectrum $\Sigma_{\alpha,V}$ has a natural covering structure. This structure makes it possible to define the so-called pre-dimensions $s_*(\alpha, V)$ and $s^*(\alpha, V)$. Then it is proven in [27, 19, 28] that

$$\dim_H \Sigma_{\alpha,V} = s_*(\alpha, V) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\alpha,V} = s^*(\alpha, V).$$

Moreover there exist two constants $0 < \rho_*(\alpha) \leq \rho^*(\alpha)$ such that

$$(1.4) \quad \lim_{V \rightarrow \infty} s_*(\alpha, V) \ln V = \rho_*(\alpha) \quad \text{and} \quad \lim_{V \rightarrow \infty} s^*(\alpha, V) \ln V = \rho^*(\alpha).$$

It is proven in [28] that $s_*(\alpha, V)$ and $s^*(\alpha, V)$ are Lipschitz continuous on any bounded interval of $[24, \infty)$.

Recently several works deal with some sub-classes of Sturm Hamiltonian. Girand [23] considered the frequency α for which the related potential can also be generated by a primitive invertible substitution. Mei [32] considered the frequency α which has eventually periodic continued fraction expansion, a strictly larger class than that considered by Girand. In both papers they showed that $\lim_{V \rightarrow 0} d_V(\alpha) = \lim_{V \rightarrow 0} s_V(\alpha) = 1$, $\mathcal{N}_{\alpha,V}$ is exact dimensional and $d_V(\alpha) < s_V(\alpha)$ for small V . This generalizes the results in [12]. Munger [34] considered the frequency of *constant type*, i.e. $\alpha_\kappa = [\kappa; \kappa, \kappa, \dots]$. He gave estimations on the optimal Hölder exponent $\gamma_V(\alpha_\kappa)$ and showed the following asymptotic formula:

$$(1.5) \quad \lim_{V \rightarrow \infty} \gamma_V(\alpha_\kappa) \ln V = \begin{cases} \frac{3}{2} \ln \alpha_1 & \kappa = 1 \\ \frac{2}{\kappa} \ln \alpha_\kappa & \kappa \geq 2. \end{cases}$$

In this paper we will consider the frequency of eventually constant type. Fix $\kappa \in \mathbb{N}$, define

$$\mathcal{F}_\kappa := \{\alpha : \alpha \text{ has expansion } [a_0; a_1, \dots, a_n, \kappa, \kappa, \dots]; a_0 \geq 0; a_i \in \mathbb{N}, 1 \leq i \leq n; n \in \mathbb{N}\}.$$

Notice that $\alpha_\kappa \in \mathcal{F}_\kappa$ and $\alpha_1 \in \mathcal{F}_1$ is the Golden number $(\sqrt{5}+1)/2$. Define $\mathcal{F} := \bigcup_{\kappa=1}^\infty \mathcal{F}_\kappa$. Any $\alpha \in \mathcal{F}$ is called a frequency of *eventually constant type*. We will study the spectral property of the related Sturm Hamiltonian for large coupling constant V .

Given a probability measure μ defined on a compact metric space X . Fix $x \in X$, we define the *local upper* and *lower* dimensions of μ at x as

$$\overline{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\ln \mu(B(x, r))}{\ln r} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\ln \mu(B(x, r))}{\ln r}.$$

In the case $\bar{d}_\mu(x) = \underline{d}_\mu(x)$, we say that the *local dimension* of μ at x exists and we denote it by $d_\mu(x)$. The *optimal Hölder exponent* γ_μ of μ is defined as

$$(1.6) \quad \gamma_\mu := \inf\{\underline{d}_\mu(x) : x \in X\}.$$

The Hausdorff dimension of μ is defined as

$$(1.7) \quad \dim_H \mu := \sup\{s : \underline{d}_\mu(x) \geq s \text{ for } \mu \text{ a.e. } x \in X\}.$$

If there exists a constant d such that $d_\mu(x) = d$ for μ a.e. $x \in X$, then necessarily $\dim_H \mu = d$. In this case we call μ *exact dimensional*.

Our main result is the following.

Theorem 1. *Fix $V > 20$ and $\kappa \in \mathbb{N}$. Then*

(i) *There exist three positive numbers $\gamma_V(\kappa)$, $d_V(\kappa)$ and $s_V(\kappa)$ such that for any $\alpha \in \mathcal{F}_\kappa$,*

$$(1.8) \quad \gamma_V(\alpha) = \gamma_V(\kappa), \quad d_V(\alpha) = d_V(\kappa) \quad \text{and} \quad s_V(\alpha) = s_V(\kappa).$$

Moreover for any fixed $\alpha \in \mathcal{F}_\kappa$, the following two assertions hold:

(ii) *$\mathcal{H}^{s_V(\kappa)}|_{\Sigma_{\alpha,V}}$ is a Gibbs type measure. Consequently, $\mathcal{H}^{s_V(\kappa)}|_{\Sigma_{\alpha,V}}$ is exact dimensional and $0 < \mathcal{H}^{s_V(\kappa)}(\Sigma_{\alpha,V}) < \infty$.*

(iii) *$\mathcal{N}_{\alpha,V}$ is a Markov measure and a Gibbs type measure. Consequently, it is exact dimensional.*

(iv) *For each $\kappa \in \mathbb{N}$, there exist three constants $0 < \hat{\varrho}_\kappa \leq \varrho_\kappa \leq \rho_\kappa$ such that*

$$\lim_{V \rightarrow \infty} \gamma_V(\kappa) \ln V = \hat{\varrho}_\kappa, \quad \lim_{V \rightarrow \infty} d_V(\kappa) \ln V = \varrho_\kappa \quad \text{and} \quad \lim_{V \rightarrow \infty} s_V(\kappa) \ln V = \rho_\kappa.$$

Moreover, $\hat{\varrho}_2 = \varrho_2 = \rho_2 = \ln(1 + \sqrt{2})$ and $\hat{\varrho}_\kappa < \varrho_\kappa < \rho_\kappa$ when $\kappa \neq 2$.

(v) *When $\kappa \neq 2$, there exists $V_0(\kappa) > 20$ such that for all $V \geq V_0(\kappa)$ we have*

$$\gamma_V(\kappa) < d_V(\kappa) < s_V(\kappa).$$

Remark 1. (1) (1.8) shows that the three quantities $\gamma_V(\alpha)$, $d_V(\alpha)$ and $s_V(\alpha)$ only depend on the “tail” of the expansion of α for α of eventually constant type. Although this is expected intuitively, the proof is far from trivial.

(2) See Theorem 5 (i) for the definition of Gibbs measure. See Definition 3 (Section 4) for the formal definition of Gibbs type measure. Roughly speaking “Gibbs type measure” means that it can be decomposed to finite pieces such that each piece is strongly equivalent to an image of a Gibbs measure under a bi-Lipschitz map (see Section 2.4 for the definition of strongly equivalence of measures). The assertions (ii) and (iii) tell us that both measures $\mathcal{H}^{s_V(\kappa)}|_{\Sigma_{\alpha,V}}$ and $\mathcal{N}_{\alpha,V}$ have very good dynamical structure.

(3) For $\alpha = \alpha_\kappa$, the first equality in (iv) is (1.5) (see [34]); for any irrational α , the third equality in (iv) is (1.4) (see [27, 19, 28]). We state them here for comparison. Our method gives a new proof for (1.5).

(4) Frequency α_1 corresponds to the Fibonacci Hamiltonian. By Remark 6, 7 and 8 in Section 8, we have

$$(1.9) \quad \hat{\varrho}_1 = \frac{3}{2} \ln \alpha_1, \quad \varrho_1 = \frac{5 + \sqrt{5}}{4} \ln \alpha_1 \quad \text{and} \quad \rho_1 = \ln(1 + \sqrt{2}).$$

The first and the third equalities in (1.9) are known, which are (1.3) and (1.2), respectively, see [9, 13]. But the second one in (1.9) is new.

(5) Recall that in [12], $d_V(1) < s_V(1)$ is proven for V small. Here for any $\kappa \neq 2$ and V is large, we show that $d_V(\kappa) < s_V(\kappa)$. As explained in [12], $\mathcal{N}_{\alpha,V}$ is the harmonic measure determined by $\Sigma_{\alpha,V}$. It is a general belief that if a planar set E is dynamically defined, then the Hausdorff dimension of the harmonic measure determined by E is strictly less than the Hausdorff dimension of E (see for example the survey paper [31] or the book [21]). Our results verify this belief in this special situation.

(6) In the case $\kappa = 2$, α_2 is sometimes called *silver* number. We call $\alpha \in \mathcal{F}_2$ a *silver type* number. By the explanation above, we still expect $d_V(2) < s_V(2)$. However, since $\varrho_2 = \rho_2$, we can not achieve this by the asymptotic formulas anymore. Finer estimations are needed. This makes the silver type number case an interesting object to study. We note that in [16], the trace map related to silver number has been studied. They showed that the non-wandering set of this map is hyperbolic if the coupling is sufficiently large. We also remark that, it follows from the general theories developed in [5] that the non-wandering set is hyperbolic for all coupling constants.

(7) Let us say a few words about the proof. Based on the analysis in [37, 29, 19] about the nested structure of the spectrum, we can introduce an auxiliary symbolic dynamical system which codes certain subsets of the spectrum. By introducing two potentials (one is additive, which is related to the density of states measure $\mathcal{N}_{\alpha,V}$; the other is almost additive, which is related to the Hausdorff measure restricted to the spectrum $\Sigma_{\alpha,V}$), we can use the thermodynamical and multifractal formalisms to analyze the spectrum and $\mathcal{N}_{\alpha,V}$, respectively. We show that both measures have the Gibbs property, moreover $\mathcal{H}^{s_V(\alpha)}|_{\Sigma_{\alpha,V}}$ is kind of measure of maximal dimension (see Section 4, Remark 3 and Theorem 9) and $\mathcal{N}_{\alpha,V}$ is kind of measure of maximal entropy (see Section 5, Theorem 10 and Theorem 11). These good structures in turn give very exact and explicit informations about the dimensions of the spectrum and the density of states measure.

Shortly after we finished the first version of our paper, we saw the impressive paper [14]. In [14], Damanik, Gorodetski and Yessen completed the picture for the Fibonacci Hamiltonian (the case $\alpha = \alpha_1$) by getting rid of the smallness and largeness restriction on the coupling constant V , giving the explicit formulas for $\gamma_V(\alpha_1), d_V(\alpha_1), s_V(\alpha_1)$ (and another important quantity— transport exponent) and the exact asymptotic behaviors of these quantities (among many other things). It seems interesting to make some comments on their methods and point out some connections of their work with ours. In their paper, they consider the so-called Fibonacci trace map T_V and the related maximal hyperbolic invariant set Λ_V . It is well known that the spectrum is obtained by intersecting a special line ℓ_V with the stable lamination of Λ_V . Through the previous works of the authors [10, 11, 12], to obtain the spectral properties of the spectrum, a crucial step is to show

that ℓ_V intersects transversally with the stable lamination of Λ_V for any $V > 0$. In [14], they made a decisive progress and succeeded to achieve this step. Then by using powerful tools from hyperbolic dynamical system, they derived the complete picture for Fibonacci Hamiltonian. Now let us point out some connections of their work with ours. In [14] Theorem 1.6, they got dimension formulas for $\Sigma_{\alpha_1, V}$ (formula (10)) and $\mathcal{N}_{\alpha_1, V}$ (formula (11)) and formula for the optimal Hölder exponent of $\mathcal{N}_{\alpha_1, V}$ (formula (12)). In our setting the counterparts are (4.7), (5.8) and (6.10). In [14] Theorem 1.10, they got exact asymptotic formulas for $\gamma_V(\alpha_1), d_V(\alpha_1), s_V(\alpha_1)$. In our case this is related to Remark 1 (4).

The rest of the paper are organized as follows. In Section 2, we introduce the notations and summarize known results and connections which we will use. In Section 3, we prove a geometric lemma, which will be used to establish a bi-Lipschitz equivalence between the dynamical subset of the spectrum and the symbolic space. In Section 4, we study the Hausdorff dimension and Hausdorff measure of a dynamical subset of the spectrum. In Section 5, we study the dimension properties of the density of states measure $\mathcal{N}_{\alpha, V}$ restricted to a dynamical subset. In Section 6, we conduct the multifractal analysis of $\mathcal{N}_{\alpha, V}$, in particular, we get an expression for the optimal Hölder exponent of $\mathcal{N}_{\alpha, V}$. In Section 7, by comparing two arbitrary dynamical subsets, we obtain the global picture. In particular, we prove Theorem 1 (i), (ii) and (iii). In Section 8, we study the asymptotic properties and prove Theorem 1 (iv) and (v). Finally in Section 9, we give another proof of the fact that $d_V(\kappa) < s_V(\kappa)$ when $\kappa \neq 2$.

2. PRELIMINARIES

In this section we summarize known results and connections which we will use.

At first we discuss the structure of the spectrum and give a coding for it. Next we collect some useful facts about Sturm Hamiltonian. Finally we recall the thermodynamical and multifractal formalisms for the almost additive potentials.

2.1. The structure of the spectrum.

We describe the structure of the spectrum $\Sigma_{\alpha, V}$ for fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $V > 0$. We also collect some known facts that will be used later, for more details, we refer to [4, 37, 29, 40].

Since $\Sigma_{\alpha, V}$ is independent with the phase ϕ , in the rest of the paper we can and will take $\phi = 0$. Assume α has continued fraction expansion $[a_0; a_1, a_2, \dots]$. Let $p_n/q_n (n > 0)$ be the n -th partial quotient of $\{\alpha\}$, the fractional part of α , given by:

$$(2.1) \quad \begin{aligned} p_{-1} &= 1, & p_0 &= 0, & p_{n+1} &= a_{n+1}p_n + p_{n-1}, & n &\geq 0, \\ q_{-1} &= 0, & q_0 &= 1, & q_{n+1} &= a_{n+1}q_n + q_{n-1}, & n &\geq 0. \end{aligned}$$

Let $n \geq 1$ and $x \in \mathbb{R}$, the transfer matrix $M_n(x)$ over q_n sites is defined by

$$\mathbf{M}_n(x) := \begin{bmatrix} x - v_{q_n} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - v_{q_{n-1}} & -1 \\ 1 & 0 \end{bmatrix} \dots \begin{bmatrix} x - v_1 & -1 \\ 1 & 0 \end{bmatrix},$$

where v_n is defined in (1.1). By convention we take

$$\mathbf{M}_{-1}(x) = \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_0(x) = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}.$$

For $n \geq 0$, $p \geq -1$, let $t_{(n,p)}(x) = \text{tr } \mathbf{M}_{n-1}(x) \mathbf{M}_n^p(x)$ and

$$\sigma_{(n,p)} = \{x \in \mathbb{R} : |t_{(n,p)}(x)| \leq 2\},$$

where $\text{tr } M$ stands for the trace of the matrix M . Then

$$(\sigma_{(n+2,0)} \cup \sigma_{(n+1,0)}) \subset (\sigma_{(n+1,0)} \cup \sigma_{(n,0)}).$$

Moreover

$$\Sigma_{\alpha,V} = \bigcap_{n \geq 0} (\sigma_{(n+1,0)} \cup \sigma_{(n,0)}).$$

The intervals in $\sigma_{(n,p)}$ are called *bands*. Take some band $B \in \sigma_{(n,p)}$, then $t_{(n,p)}(x)$ is monotone on B and $t_{(n,p)}(B) = [-2, 2]$. We call $t_{(n,p)}$ the *generating polynomial* of B and denote it by $h_B := t_{(n,p)}$.

$\{\sigma_{(n+1,0)} \cup \sigma_{(n,0)} : n \geq 0\}$ form a covering of $\Sigma_{\alpha,V}$. However there are some repetitions between $\sigma_{(n,0)} \cup \sigma_{(n-1,0)}$ and $\sigma_{(n+1,0)} \cup \sigma_{(n,0)}$. It is possible to choose a covering of $\Sigma_{\alpha,V}$ elaborately such that we can get rid of these repetitions, as we will describe in the follows:

Definition 1. ([37, 29]) For $V > 4$, $n \geq 0$, we define three types of bands as follows:

- (n, I)-type band: a band of $\sigma_{(n,1)}$ contained in a band of $\sigma_{(n,0)}$;
- (n, II)-type band: a band of $\sigma_{(n+1,0)}$ contained in a band of $\sigma_{(n,-1)}$;
- (n, III)-type band: a band of $\sigma_{(n+1,0)}$ contained in a band of $\sigma_{(n,0)}$.

All three types of bands actually occur and they are disjoint. We call these bands *spectral generating bands of order n* . Note that there are only two spectral generating bands of order 0, one is $\sigma_{(0,1)} = [V-2, V+2]$ with generating polynomial $t_{(0,1)} = x - V$ and type (0, I), the other is $\sigma_{(1,0)} = [-2, 2]$ with generating polynomial $t_{(1,0)} = x$ and type (0, III). They are contained in $\sigma_{(0,0)} = (-\infty, +\infty)$ with generating polynomial $t_{(0,0)} \equiv 2$. For convenience, we call $\sigma_{(0,0)}$ the spectral generating band of order -1 .

For any $n \geq -1$, denote by \mathcal{B}_n the set of spectral generating bands of order n , then the intervals in \mathcal{B}_n are disjoint. Moreover ([37, 29])

- $(\sigma_{(n+2,0)} \cup \sigma_{(n+1,0)}) \subset \bigcup_{B \in \mathcal{B}_n} B \subset (\sigma_{(n+1,0)} \cup \sigma_{(n,0)})$, thus

$$\Sigma_{\alpha,V} = \bigcap_{n \geq 0} \bigcup_{B \in \mathcal{B}_n} B;$$

- any (n, I)-type band contains only one band in \mathcal{B}_{n+1} , which is of ($n+1, II$)-type.
- any (n, II)-type band contains $2a_{n+1} + 1$ bands in \mathcal{B}_{n+1} , $a_{n+1} + 1$ of which are of ($n+1, I$)-type and a_{n+1} of which are of ($n+1, III$)-type.
- any (n, III)-type band contains $2a_{n+1} - 1$ bands in \mathcal{B}_{n+1} , a_{n+1} of which are of ($n+1, I$)-type and $a_{n+1} - 1$ of which are of ($n+1, III$)-type.

Thus $\{\mathcal{B}_n\}_{n \geq 0}$ forms a natural covering([30, 27]) of the spectrum $\Sigma_{\alpha,V}$.

2.2. The coding of the spectrum.

In the following we give a coding of the spectrum $\Sigma_{\alpha,V}$ based on [28], however, we note that the presentation here is slightly different from that in [28].

For each $N \in \mathbb{N}$, we define an alphabet \mathcal{A}_N as

$$\mathcal{A}_N := \{(I, j)_N : j = 1, \dots, N+1\} \cup \{(II, 1)_N\} \cup \{(III, j)_N : j = 1, \dots, N\}.$$

Then $\#\mathcal{A}_N = 2N + 2$. We order the elements in \mathcal{A}_N as

$$(I, 1)_N < \dots < (I, N+1)_N < (II, 1)_N < (III, 1)_N < \dots < (III, N)_N.$$

To simplify the notation we rename the above line as

$$e_{N,1} < e_{N,2} < \dots < e_{N,2N+2}.$$

Given $e_{N,i} \in \mathcal{A}_N$ and $e_{M,j} \in \mathcal{A}_M$, we call $e_{N,i}e_{M,j}$ *admissible*, denote by $e_{N,i} \rightarrow e_{M,j}$, if

$$\begin{aligned} (e_{N,i}, e_{M,j}) \in & \{((I, k)_N, (II, 1)_M) : 1 \leq k \leq N+1\} \cup \\ & \{((II, 1)_N, (I, l)_M) : 1 \leq l \leq M+1\} \cup \\ & \{((II, 1)_N, (III, l)_M) : 1 \leq l \leq M\} \cup \\ & \{((III, k)_N, (I, l)_M) : 1 \leq k \leq N, 1 \leq l \leq M\} \cup \\ & \{((III, k)_N, (III, l)_M) : 1 \leq k \leq N, 1 \leq l \leq M-1\}. \end{aligned}$$

For pair $(\mathcal{A}_N, \mathcal{A}_M)$, we define the incidence matrix $A_{NM} = (a_{ij})$ of size $(2N+2) \times (2M+2)$ as

$$a_{ij} = \begin{cases} 1 & e_{N,i} \rightarrow e_{M,j} \\ 0 & \text{otherwise} \end{cases}$$

When $N = M$, we write $A_N := A_{NN}$. For any $N \in \mathbb{N}$, we define a related matrix as follows:

$$\hat{A}_N = \begin{bmatrix} 0 & 1 & 0 \\ N+1 & 0 & N \\ N & 0 & N-1 \end{bmatrix}$$

We also define $\mathcal{A}_0 := \{I, III\}$. Write $e_1 = I$ and $e_2 = III$. Given $e_i \in \mathcal{A}_0$ and $e_{M,j} \in \mathcal{A}_M$, we call $e_i e_{M,j}$ *admissible*, denote by $e_i \rightarrow e_{M,j}$, if

$$\begin{aligned} (e_i, e_{M,j}) \in & \{(I, (II, 1)_M)\} \cup \{(III, (I, l)_M) : 1 \leq l \leq M\} \cup \\ & \{(III, (III, l)_M) : 1 \leq l \leq M-1\}. \end{aligned}$$

For pair $(\mathcal{A}_0, \mathcal{A}_M)$, we define the incidence matrix $A_{0M} = (a_{ij})$ of size $2 \times (2M+2)$ as

$$a_{ij} = \begin{cases} 1 & e_i \rightarrow e_{M,j} \\ 0 & \text{otherwise} \end{cases}$$

Recall that $\alpha = [a_0; a_1, a_2, \dots]$. Define a symbolic space $\Omega^{(\alpha)}$ with alphabet sequence $\{\mathcal{A}_0\} \cup \{\mathcal{A}_{a_i} : i \geq 1\}$ and incidence matrix sequence $\{A_{0a_1}\} \cup \{A_{a_i a_{i+1}} : i \geq 1\}$ as

$$\Omega^{(\alpha)} = \{e_{i_0} e_{a_1, i_1} e_{a_2, i_2} \dots : e_{i_0} \in \mathcal{A}_0; e_{a_j, i_j} \in \mathcal{A}_{a_j}; e_{i_0} \rightarrow e_{a_1, i_1}, e_{a_j, i_j} \rightarrow e_{a_{j+1}, i_{j+1}}, j \geq 1\}.$$

For $\omega \in \Omega^{(\alpha)}$, we write $\omega|_n = \omega_0 \cdots \omega_n$. More generally we write $\omega[n, \cdots, n+k]$ for $\omega_n \cdots \omega_{n+k}$. Define $\Omega_n^{(\alpha)} := \{\omega|_n : \omega \in \Omega^{(\alpha)}\}$ and $\Omega_*^{(\alpha)} = \bigcup_n \Omega_n^{(\alpha)}$. Given $w = w_0 \cdots w_n \in \Omega_n^{(\alpha)}$, denote by $|w|$ the length of w , then $|w| = n+1$. If $w_n = (T, j)_{a_n}$, then we write $t_w = T$, and call w has *type* T . If $w = uw'$, then we say u is a *prefix* of w and denote by $u \prec w$. Given $u = u_0 \cdots u_n \in \Omega_n^{(\alpha)}$ and $v = v_0 \cdots v_m \in \prod_{j=n}^{n+m} \mathcal{A}_{a_j}$, if $u_n = v_0$, then we write $u \star v := u_0 \cdots u_n v_1 \cdots v_m$. For $\omega, \tilde{\omega} \in \Omega^{(\alpha)}$, we denote by $\omega \wedge \tilde{\omega}$ the maximal common prefix of ω and $\tilde{\omega}$. Given $w \in \Omega_n^{(\alpha)}$, we define the cylinder

$$[w] := \{\omega \in \Omega^{(\alpha)} : \omega|_n = w\}.$$

In the following we explain that $\Omega^{(\alpha)}$ is a coding of the spectrum $\Sigma_{\alpha, V}$. Define B_I to be the unique $(0, I)$ type band in \mathcal{B}_0 and B_{III} to be the unique $(0, III)$ type band in \mathcal{B}_0 . Then

$$\mathcal{B}_0 = \{B_w : w \in \Omega_0^{(\alpha)}\}.$$

Assume B_w is defined for any $w \in \Omega_{n-1}^{(\alpha)}$. Now for any $w \in \Omega_n^{(\alpha)}$, write $w = w'e = w'(T, j)_{a_n}$. Then define B_w to be the unique j -th (n, T) type band in \mathcal{B}_n which is contained in $B_{w'}$. Then

$$\mathcal{B}_n = \{B_w : w \in \Omega_n^{(\alpha)}\}.$$

Thus we can define a natural projection $\pi : \Omega^{(\alpha)} \rightarrow \Sigma_{\alpha, V}$ as

$$\pi(\omega) := \bigcap_{n \geq 0} B_{\omega|_n}.$$

It is seen that π is a one-to-one map. So $\Omega^{(\alpha)}$ is a coding of $\Sigma_{\alpha, V}$. Write $X_w = \pi([w])$ for each $w \in \Omega_*^{(\alpha)}$ and denote by h_w the generating polynomial of B_w .

For any $n \geq 1$, we also define the following symbolic space:

$$\Omega^{(\alpha, n)} = \{e_{a_n, i_n} e_{a_{n+1}, i_{n+1}} \cdots : e_{a_j, i_j} \in \mathcal{A}_{a_j}; e_{a_j, i_j} \rightarrow e_{a_{j+1}, i_{j+1}}, j \geq n\}.$$

Formally, we obtain $\Omega^{(\alpha, n)}$ by shifting $\Omega^{(\alpha)}$ to the left n times. In general, all the $\Omega^{(\alpha, n)}$ are different, thus it is hard to define a dynamic on them. However, if α is of eventually constant type and has continued fraction expansion $[a_0; a_1, \cdots, a_{\hat{n}}, \kappa, \kappa, \cdots]$, then it is easy to see that all the $\{\Omega^{(\alpha, n)} : n > \hat{n}\}$ are the same and indeed they are a subshift of finite type, with alphabet \mathcal{A}_κ and incidence matrix $A_\kappa = A_{\kappa\kappa}$. For this reason, we give the following definition:

Definition 2. For any $\kappa \in \mathbb{N}$, we denote the subshift of finite type with alphabet \mathcal{A}_κ and incidence matrix A_κ by $\Omega^{(\kappa)}$. Together with the shift map σ , $(\Omega^{(\kappa)}, \sigma)$ becomes a dynamical system.

By endowed with the usual metric on $\Omega^{(\kappa)}$, $(\Omega^{(\kappa)}, \sigma)$ becomes a topological dynamical system. The following lemma implies that indeed $(\Omega^{(\kappa)}, \sigma)$ is topologically mixing.

Recall that a nonnegative square matrix B is called *primitive* if there exists some $k \in \mathbb{N}$ such that all the entries of B^k are positive.

Lemma 1. *For each $\kappa \in \mathbb{N}$, A_κ, \hat{A}_κ are primitive and have the same Perron-Frobenius eigenvalue α_κ . Moreover \hat{A}_κ has three different eigenvalues $\alpha_\kappa, -1, -1/\alpha_\kappa$, consequently \hat{A}_κ is diagonalizable.*

Proof. It is straightforward to check that $\hat{A}_\kappa^5 > 0$, thus \hat{A}_κ is primitive. We also have

$$\det(\lambda I_3 - \hat{A}_\kappa) = (\lambda^2 - \kappa\lambda - 1)(\lambda + 1).$$

Thus the Perron-Frobenius eigenvalue of \hat{A}_κ is α_κ and \hat{A}_κ has three different eigenvalues $\alpha_\kappa, -1, -1/\alpha_\kappa$.

On the other hand if we consider the graph related to the incidence matrix A_κ , then it is easy to show that the graph is aperiodic. Consequently A_κ is primitive. By direct computation we get that

$$\det(\lambda I_{2\kappa+2} - A_\kappa) = \lambda^{2\kappa-1}(\lambda^2 - \kappa\lambda - 1)(\lambda + 1).$$

Thus the Perron-Frobenius eigenvalue of A_κ is also α_κ . \square

In what follows, we will make essential use of this symbolic dynamic $(\Omega^{(\kappa)}, \sigma)$ to understand the structure of the spectrum and the density of states measure of Sturm Hamiltonian with frequencies of eventually constant type.

2.3. Useful results for Sturm Hamiltonian.

In this subsection we collect some useful results for Sturm Hamiltonian.

Fix an irrational frequency α with continued fraction expansion $[a_0; a_1, a_2, \dots]$ and consider the operator $H_{\alpha, V, 0}$. We write $h_k(x) := t_{k+1, 0}(x) = \text{tr } M_k(x)$ and write

$$(2.2) \quad \sigma_k := \sigma_{(k+1, 0)} = \{x \in \mathbb{R} : |h_k(x)| \leq 2\}.$$

The following two lemmas are essentially proven in [37]:

Lemma 2. $\sigma_k = \{B \in \mathcal{B}_k : B \text{ is of type } (k, II) \text{ or } (k, III)\}.$

Define $v^I = (1, 0, 0), v^{II} = (0, 1, 0), v^{III} = (0, 0, 1)$ and $v_* = (0, 1, 1)^T$.

Lemma 3. *Given $w \in \Omega_m^{(\alpha)}$, then*

$$(2.3) \quad \#\{u \in \Omega_{m+n}^{(\alpha)} : w \prec u, t_u = II, III\} = v^{t_w} \cdot \hat{A}_{a_{m+1}} \cdots \hat{A}_{a_{m+n}} \cdot v_*.$$

The following theorem is [19] Theorem 2.1, see also [28] Theorem 3.1.

Theorem 2 (Bounded variation). *Let $V > 20$ and $\alpha = [a_0; a_1, a_2, \dots]$ be irrational with a_n bounded by M . Then there exists a constant $\xi = \xi(V, M) > 1$ such that for any spectral generating band B with generating polynomial h ,*

$$\xi^{-1} \leq \left| \frac{h'(x_1)}{h'(x_2)} \right| \leq \xi, \quad \forall x_1, x_2 \in B.$$

The following theorem is [19] Theorem 5.1, see also [28] Theorem 3.3.

Theorem 3 (Bounded covariation). *Let $V > 20$, $\alpha = [a_0; a_1, a_2, \dots]$, $\tilde{\alpha} = [\tilde{a}_0; \tilde{a}_1, \tilde{a}_2, \dots]$ be irrational with a_n, \tilde{a}_n bounded by M . Then there exists constant $\eta = \eta(V, M) > 1$ such that if $w, wu \in \Omega_*^{(\alpha)}$ and $\tilde{w}, \tilde{w}u \in \Omega_*^{(\tilde{\alpha})}$, then*

$$\eta^{-1} \frac{|B_{wu}|}{|B_w|} \leq \frac{|B_{\tilde{w}u}|}{|B_{\tilde{w}}|} \leq \eta \frac{|B_{wu}|}{|B_w|}.$$

We remark that in [19], they only considered the case $\alpha = \tilde{\alpha}$. However by checking the proof, the same argument indeed shows the stronger result as stated in Theorem 3.

The following lemma is a direct consequence of [19] Corollary 3.1:

Lemma 4. *Let $V > 20$ and $\alpha = [a_0; a_1, a_2, \dots]$ be irrational with a_n bounded by M . Then there exist constants $0 < c_1 = c_1(V, M) < c_2 = c_2(V, M) < 1$ and $n_0 = n_0(V, M) \in \mathbb{N}$ such that for any $n > n_0$ and any $w \in \Omega_n^{(\alpha)}$*

$$c_1 |B_{w|_{n-n_0}}| \leq |B_w| \leq c_2 |B_{w|_{n-n_0}}|.$$

There exists constant $0 < c_3 = c_3(V, M) < 1$ such that for any $w \in \Omega_n^{(\alpha)}$

$$(2.4) \quad c_3^n \leq |B_w| \leq 2^{2-n}.$$

The following lemma is [28] Lemma 3.7 by taking $a_i = \kappa$.

Lemma 5. *Assume $V > 20$ and $\alpha = \alpha_\kappa$. Write $t_1 = (V - 8)/3$ and $t_2 = 2(V + 5)$. For $e_{\kappa, \kappa+2} = (II, 1)_\kappa$ and $w = w_0 \cdots w_n \in \Omega_n^{(\alpha_\kappa)}$, write $|w|_{e_{\kappa, \kappa+2}} := \#\{1 \leq i \leq n : w_i = e_{\kappa, \kappa+2}\}$. Then*

$$t_2^{(1-\kappa)|w|_{e_{\kappa, \kappa+2}}} (t_2 \kappa^3)^{|w|_{e_{\kappa, \kappa+2}} - n} \leq |B_w| \leq 4t_1^{(1-\kappa)|w|_{e_{\kappa, \kappa+2}}} (t_1 \kappa)^{|w|_{e_{\kappa, \kappa+2}} - n}.$$

2.4. Recall on thermodynamical formalism and multifractal analysis.

If X is a compact metric space and $T : X \rightarrow X$ is continuous, then (X, T) is called a *topological dynamical system*, TDS for short. $\mathcal{M}(X)$ is the set of all probability measures supported on X . $\mathcal{M}(X, T)$ is the set of all T -invariant probability measures supported on X . Given $\mu, \nu \in \mathcal{M}(X)$, if there exists a constant $C > 1$ such that $C^{-1}\nu \leq \mu \leq C\nu$, then we say μ and ν are *strongly equivalent* and denote by $\mu \asymp \nu$.

Assume (X, T) is a TDS and $\Phi = \{\phi_n : n \geq 0\}$ is a family of continuous functions from X to \mathbb{R} . If there exists a constant $C(\Phi) \geq 0$ such that for any $n, k \geq 0$

$$|\phi_{n+k}(x) - \phi_n(x) - \phi_k(T^n x)| \leq C(\Phi),$$

then Φ is called a family of *almost additive* potentials. We use $C_{aa}(X, T)$ to denote the set of all the almost additive potentials defined on X . When $C(\Phi) = 0$, it is seen that $\phi_n(x) = \sum_{j=0}^{n-1} \phi_1(T^j x) =: S_n \phi_1(x)$. That is, ϕ_n is the *ergodic sum* of ϕ_1 . In this case we say Φ is a family of *additive* potentials.

Given $\Phi \in C_{aa}(X, T)$. If there exists a constant $c > 0$ such that $\phi_n(x) \geq cn$ for any $n \geq 0$, then we say Φ is *positive* and write $\Phi \in C_{aa}^+(X, T)$. Similarly, if there exists a constant $c < 0$ such that $\phi_n(x) \leq cn$ for any $n \geq 0$, then we say Φ is *negative* and write $\Phi \in C_{aa}^-(X, T)$.

Given $\Phi \in C_{aa}(X, T)$, by subadditivity, $\Phi_*(\mu) := \lim_{n \rightarrow \infty} \int_X \frac{\phi_n}{n} d\mu$ exists for every $\mu \in \mathcal{M}(X, T)$. Notice that if Φ is positive(negative), then $\Phi_*(\mu) > 0 (< 0)$.

Given a subshift of finite type (Σ_A, σ) and $\Phi \in C_{aa}(\Sigma_A, \sigma)$. If there exists constant $D(\Phi) \geq 0$ such that for any $n \geq 0$,

$$\sup\{|\phi_n(x) - \phi_n(y)| : x|_n = y|_n\} \leq D(\Phi)$$

Then we say Φ has *bounded variation* property.

2.4.1. Thermodynamical formalism.

Given $\Phi \in C_{aa}(\Sigma_A, \sigma)$, the topological pressure is defined as

$$P(\Phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \ln \sum_{|w|=n} \exp(\sup_{x \in [w]} \phi_n(x)).$$

The following extension of the classical variational principle holds:

Theorem 4. [2, 33, 7, 3] *Let (Σ_A, σ) be a topologically mixing subshift of finite type. For any $\Phi \in C_{aa}(\Sigma_A, \sigma)$ we have*

$$(2.5) \quad P(\Phi) = \sup\{h_\mu(T) + \Phi_*(\mu) : \mu \in \mathcal{M}(\Sigma_A, \sigma)\}.$$

Combining with the monotonicity of pressure, this variational principle has the following consequence:

Corollary 1. *For any $\Phi, \Psi \in C_{aa}(\Sigma_A, \sigma)$, the function $Q(s) := P(\Phi + s\Psi)$ is convex on \mathbb{R} . Consequently Q is continuous. If moreover $\Psi \in C_{aa}^-(\Sigma_A, \sigma)$, then $Q(s)$ is strictly decreasing and*

$$\lim_{s \rightarrow -\infty} Q(s) = \infty \quad \text{and} \quad \lim_{s \rightarrow \infty} Q(s) = -\infty.$$

Thus $Q(s) = 0$ has a unique solution.

Theorem 5. [2, 33] *Assume $\Phi \in C_{aa}(\Sigma_A, \sigma)$ has bounded variation property. Then*

(i) *There exists a unique invariant measure μ_Φ such that*

$$C^{-1} \leq \frac{\mu_\Phi([w])}{\exp(-nP(\Phi) + \phi_n(x))} \leq C \quad (\forall x \in [w]).$$

μ_Φ is called the *Gibbs measure* related to Φ .

(ii) *If $\tilde{\Phi} \in C_{aa}(\Sigma_A, \sigma)$ and $D > 0$ is a constant such that $\|\phi_n - \tilde{\phi}_n\| \leq D$ for any $n \geq 0$, then $\tilde{\Phi}$ also has bounded variation property. Moreover $P(\Phi) = P(\tilde{\Phi})$ and $\mu_\Phi = \mu_{\tilde{\Phi}}$.*

(iii) μ_Φ is strongly mixing and is the unique invariant measure which attains the supremum of (2.5).

If $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ such that $P(\Phi) = h_\mu(\sigma) + \Phi_*(\mu)$, then μ is called an *equilibrium state* of Φ . The above proposition shows that every $\Phi \in C_{aa}(\Sigma_A, \sigma)$ with bounded variation property has a unique equilibrium state.

2.4.2. Multifractal analysis.

We recall some results proved in [1](see also [3]), which we will need in this paper.

Given a $\Psi \in C_{aa}^-(\Sigma_A, \sigma)$, define on (Σ_A, σ) a weak Gibbs metric d_Ψ as

$$(2.6) \quad d_\Psi(x, y) = \sup_{z \in [x \wedge y]} \exp(\psi_{|x \wedge y|}(z)).$$

This kind of metric is considered in [22, 25, 1]. In the following we will work on the metric space (Σ_A, d_Ψ) .

Let $\Phi \in C_{aa}(\Sigma_A, \sigma)$ and $\Theta \in C_{aa}^-(\Sigma_A, \sigma)$. For any $\beta \in \mathbb{R}$, define the level set

$$\Lambda_{\Phi/\Theta}(\beta) := \left\{ x \in \Sigma_A : \lim_{n \rightarrow \infty} \frac{\phi_n(x)}{\theta_n(x)} = \beta \right\}.$$

Since Θ is negative, $\Theta_*(\mu) < 0$ for any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. Define

$$L_{\Phi/\Theta} := \left\{ \frac{\Phi_*(\mu)}{\Theta_*(\mu)} : \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

Then $L_{\Phi/\Theta}$ is an interval. For any $q, \beta \in \mathbb{R}$ we define $\mathcal{L}_{\Phi/\Theta}(q, \beta)$ to be the unique solution $t = t(q, \beta)$ of the equation $P(q(\Phi - \beta\Theta) + t\Psi) = 0$. For any $\beta \in L_{\Phi/\Theta}$, define

$$\mathcal{L}_{\Phi/\Theta}^*(\beta) = \inf_{q \in \mathbb{R}} \mathcal{L}_{\Phi/\Theta}(q, \beta).$$

Theorem 6. [1] $\Lambda_{\Phi/\Theta}(\beta) \neq \emptyset \Leftrightarrow \beta \in L_{\Phi/\Theta}$. If $\beta \in L_{\Phi/\Theta}$, then

$$\dim_H \Lambda_{\Phi/\Theta}(\beta) = \mathcal{L}_{\Phi/\Theta}^*(\beta).$$

The following dimension formula for Gibbs measure is also useful.

Theorem 7. [3] Given $\Phi \in C_{aa}(\Sigma_A, \sigma)$ with bounded variation property. Let μ_Φ be the related Gibbs measure. Then μ_Φ is exact dimensional and

$$(2.7) \quad \dim_H \mu_\Phi = \frac{h_{\mu_\Phi}(\sigma)}{-\Psi_*(\mu)}.$$

Finally we say some words about notations. Given two positive sequences $\{a_n\}$ and $\{b_n\}$, $a_n \lesssim b_n$ means that there exists some constant $C > 0$ such that $a_n \leq Cb_n$ for all $n \in \mathbb{N}$. $a_n \sim b_n$ means that $a_n \lesssim b_n$ and $b_n \lesssim a_n$. $a_n \lesssim_{\gamma_1, \dots, \gamma_k} b_n$ means that $a_n \lesssim b_n$ with the constant C only depending on $\gamma_1, \dots, \gamma_k$.

3. A GEOMETRIC LEMMA

In this section we will prove a geometric lemma, which claims that the ratios of the lengths of a gap and the minimal band which contains it are bounded from below. This lemma is fundamental in our study on the metric property of the spectrum because through it we can establish a Lipschitz equivalence between the symbolic space and the spectrum. Thus all the dimension problems of the spectrum are completely converted to those of the symbolic space, where we have dynamical tools to use.

Fix $V > 20$ and $\alpha = [a_0; a_1, a_2, \dots]$ with $a_i \leq M$. Write $\text{Co}(\Sigma_{\alpha, V}) \setminus \Sigma_{\alpha, V} = \bigcup_i G_i$, where $\text{Co}(\Sigma_{\alpha, V})$ is the convex hull of $\Sigma_{\alpha, V}$. Each G_i is called a *gap* of the spectrum. A gap G

is called of *order* n , if G is covered by some band in \mathcal{B}_n but not covered by any band in \mathcal{B}_{n+1} . Denote by \mathcal{G}_n the set of gaps of order n . For any $G \in \mathcal{G}_n$, let B_G be the unique band in \mathcal{B}_n which contains G .

Lemma 6. *There exists a constant $C = C(M, V) \in (0, 1)$ such that for any gap $G \in \bigcup_{n \geq 0} \mathcal{G}_n$ we have $|G| \geq C|B_G|$.*

Proof. Write $q := a_{n+1}$, then $q \leq M$ by the assumption. Given $B_w \in \mathcal{B}_n$, we study the gaps of order n contained in B_w . If w has type $t_w = I$, then there exists a unique band $B_{we_q, q+2} \in \mathcal{B}_{n+1}$ which is contained in B_w , thus there is no gap of order n in B_w .

Now assume $t_w = II$. Then by [37], there exist $2q + 1$ bands of order $n + 1$, which are disjoint and ordered as follows:

$$B_{we_q, 1} < B_{we_q, q+3} < B_{we_q, 2} < B_{we_q, q+4} < B_{we_q, 3} < \cdots < B_{we_q, 2q+2} < B_{we_q, q+1}.$$

Thus there are $2q$ gaps of order n in B_w . We list them as $\{G_1, \dots, G_{2q}\}$.

Let $S_p(x)$ be the Chebyshev polynomial defined by

$$S_0(x) \equiv 0, \quad S_1(x) \equiv 1, \quad S_{p+1}(x) = xS_p(x) - S_{p-1}(x) \quad (p \geq 1).$$

It is well known that

$$(3.1) \quad S_p(2 \cos \theta) = \frac{\sin p\theta}{\sin \theta}, \quad \theta \in [0, \pi].$$

Following [19], for each $p \in \mathbb{N}$ and $1 \leq l \leq p$, we define

$$I_{p,l} := \{2 \cos \frac{l+c}{p+1} \pi : |c| \leq \frac{1}{10} \text{ and } |S_{p+1}(2 \cos \frac{l+c}{p+1} \pi)| \leq \frac{1}{4}\}.$$

It is seen that $I_{p,l}, l = 1, \dots, p$ are disjoint.

Claim: $I_{p,l}$ and $I_{p-1,s}$ are disjoint.

◁ Fix any $x \in I_{p-1,s}$, write $x = 2 \cos \theta$. Then

$$(3.2) \quad \frac{s-1/10}{p} \pi \leq \theta \leq \frac{s+1/10}{p} \pi \quad \text{and} \quad \left| \frac{\sin p\theta}{\sin \theta} \right| = |S_p(2 \cos \theta)| \leq \frac{1}{4}.$$

We need to show that $x \notin I_{p,l}$. If otherwise, by the definition of $I_{p,l}$ and (3.2) we should have

$$|\cos p\theta| - \left| \frac{\sin p\theta}{\sin \theta} \right| \leq \left| \frac{\sin(p+1)\theta}{\sin \theta} \right| = |S_{p+1}(2 \cos \theta)| \leq \frac{1}{4}.$$

Thus we have $|\cos p\theta| \leq 1/2$. On the other hand, still by (3.2) we have

$$(s-1/10)\pi \leq p\theta \leq (s+1/10)\pi.$$

Thus $|\cos p\theta| \geq \cos \frac{\pi}{10} > \frac{1}{2}$, which is a contradiction. ▷

By (3.1) and the claim above, it is easy to check that the following intervals are disjoint subintervals of $[-2, 2]$ and ordered as

$$I_{p+1,1} < I_{p,1} < I_{p+1,2} < \cdots < I_{p+1,p} < I_{p,p} < I_{p+1,p+1}.$$

There are $2p$ gaps $\{\tilde{G}_1^{(p)}, \dots, \tilde{G}_{2p}^{(p)}\}$. Define

$$g(p) := \min\{|\tilde{G}_j^{(p)}| : j = 1, \dots, 2p\} \quad \text{and} \quad g := \min\{g(1), \dots, g(M)\}.$$

Then $g > 0$ is a constant only depending on M .

Assume $x_* \in B_w$ such that

$$(3.3) \quad |h'_w(x_*)||B_w| = 4.$$

By [19] Proposition 3.1, we have

$$h_w(B_{we_{q,l}}) \subset I_{q+1,l} \quad (1 \leq l \leq q+1) \quad \text{and} \quad h_w(B_{we_{q,l+q+2}}) \subset I_{q,l} \quad (1 \leq l \leq q).$$

Since $h_w : B_w \rightarrow [-2, 2]$ is a bijection, $h_w(G_j) \supset \tilde{G}_j^{(q)}$ for each j . Fix a gap G_j in B_w , then there exists $x_j \in G_j$ such that

$$(3.4) \quad |h'_w(x_j)||G_j| = |h_w(G_j)| \geq |\tilde{G}_j^{(q)}|.$$

Recall that $q \leq M$. By Theorem 2, (3.3), (3.4), there exist a constant $c(M, V) > 0$ such that

$$\frac{|G_j|}{|B_w|} \geq \frac{g}{4} \cdot \frac{|h'_w(x_*)|}{|h'_w(x_j)|} \geq cg =: C_1(M, V) > 0.$$

If $t_w = III$, the same proof as above shows that there exists a constant $C_2(M, V) > 0$ such that for any gap G in B_w we have

$$\frac{|G|}{|B_w|} \geq C_2(M, V) > 0.$$

Let $C = \min\{C_1, C_2\}$, the result follows. \square

4. HAUSDORFF DIMENSION OF THE DYNAMICAL SUBSETS

From Section 4 to Section 6, we always fix $\kappa \in \mathbb{N}$ and $\alpha \in \mathcal{F}_\kappa$ with continued fraction expansion $[a_0; a_1, \dots, a_{\hat{n}}, \kappa, \kappa, \dots]$. In this section, we will study the Hausdorff dimension and Hausdorff measure of $\Sigma_{\alpha, V}$ by the aid of the dynamical system $(\Omega^{(\kappa)}, \sigma)$.

We have coded $\Sigma_{\alpha, V}$ by a symbolic space $\Omega^{(\alpha)}$. We also noticed that in general it is hard to define a dynamic on $\Omega^{(\alpha)}$. However since now α is of eventually constant type, if we shift $\Omega^{(\alpha)}$ \hat{n} times, we get a subshift of finite type $\Omega^{(\kappa)}$. It is this fact which makes it possible to use the machinery of thermodynamical formalism to study the Hausdorff dimensions of the spectrum, with the aid of $(\Omega^{(\kappa)}, \sigma)$.

At the technique level, our strategy is as follows: At first we introduce a family of subsets of $\Sigma_{\alpha, V}$, which we called dynamical subsets, such that each subset in this family can be coded by $(\Omega^{(\kappa)}, \sigma)$ and the union of them are the whole spectrum. With the aid of $(\Omega^{(\kappa)}, \sigma)$, we can obtain all the fractal properties of the dynamical subset. Finally we will show that all the properties keep unchanged when we exhaust all the possible choices of subsets. Thus we get a global result for the whole spectrum. We will finish the final step in Section 7.

4.1. Dynamical subsets.

Fix $N = N_\alpha \geq 4 + \hat{n}$ such that $A_\kappa^{N-\hat{n}} > 0$. Then the set of the last letters of words in $\Omega_N^{(\alpha)}$ is \mathcal{A}_κ . In other words,

$$\{w_N : w \in \Omega_N^{(\alpha)}\} = \mathcal{A}_\kappa.$$

Since κ is fixed, we write $e_i := e_{\kappa,i} \in \mathcal{A}_\kappa$ for simplicity. Define a subsets of $(\Omega_N^{(\alpha)})^{2\kappa+2}$ as

$$(4.1) \quad \mathcal{D} = \mathcal{D}(\alpha) := \{(w^{e_1}, \dots, w^{e_{2\kappa+2}}) \in (\Omega_N^{(\alpha)})^{2\kappa+2} : w_N^{e_i} = e_i, i = 1, \dots, 2\kappa+2\}.$$

We denote by $\vec{w} := (w^{e_1}, \dots, w^{e_{2\kappa+2}})$ an element in \mathcal{D} . Recall that $X_w = \pi([w])$ for each $w \in \Omega_*^{(\alpha)}$. Given $\vec{w} \in \mathcal{D}$, define

$$\Sigma_{\vec{w}} := \bigcup_{i=1}^{2\kappa+2} X_{w^{e_i}}.$$

It is seen that $\Sigma_{\vec{w}}$ is made of $2\kappa+2$ N -level basic sets of $\Sigma_{\alpha,V}$ and

$$\Sigma_{\alpha,V} = \bigcup_{\vec{w} \in \mathcal{D}} \Sigma_{\vec{w}}.$$

Now we define a projection $\pi_{\vec{w}} : \Omega^{(\kappa)} \rightarrow \Sigma_{\vec{w}}$ as

$$(4.2) \quad \pi_{\vec{w}}(\omega) = \pi(w^{\omega_0} \sigma \omega) = \pi(w^{\omega_0} \star \omega)$$

(see the definition of $v \star w$ in Section 2.2). It is ready to show that $\pi_{\vec{w}}$ is a one-to-one map. We call $\Sigma_{\vec{w}}$ a *dynamical subset* of $\Sigma_{\alpha,V}$.

We will study the dimension properties of $\Sigma_{\vec{w}}$ at first, then by comparing two different dynamical subsets $\Sigma_{\vec{v}}$ and $\Sigma_{\vec{w}}$, we obtain a global picture (we will finish this in Section 7).

From now on until the end of this section we will fix some $\vec{w} \in \mathcal{D}$.

4.2. Almost additive potentials related to Lyapunov exponents.

We will define some $\Psi \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$, which captures the exponential rate of the length of the generating bands and can be viewed as Lyapunov exponent function. We will see that Ψ is intimately related to the Hausdorff dimension of $\Sigma_{\vec{w}}$.

For each $n \in \mathbb{N}$ and each $\omega \in \Omega^{(\kappa)}$, define

$$(4.3) \quad \psi_n(\omega) := \ln |B_{w^{\omega_0} \omega[1, \dots, n]}| = \ln |B_{w^{\omega_0} \star \omega|_n}|,$$

where $|B_w|$ denote the length of B_w .

Lemma 7. $\Psi = \{\psi_n : n \geq 0\} \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$ and Ψ has bounded variation property.

Proof. Given $\omega \in \Omega^{(\kappa)}$ we have $\psi_n(\omega) = \ln |B_{w^{\omega_0} \omega[1, \dots, n]}|$, $\psi_{n+k}(\omega) = \ln |B_{w^{\omega_0} \omega[1, \dots, n+k]}|$ and $\psi_k(\sigma^n \omega) = \ln |B_{w^{\omega_n} \omega[n+1, n+k]}|$. By Theorem 3,

$$\frac{|B_{w^{\omega_0} \omega[1, \dots, n+k]}|}{|B_{w^{\omega_0} \omega[1, \dots, n]}|} \sim_{V, \alpha} \frac{|B_{w^{\omega_n} \omega[n+1, n+k]}|}{|B_{w^{\omega_n}}|}.$$

Notice that there are only finitely many different bands B_w with $|w| = N+1$ and N only depends on α , thus we conclude from the above equation that

$$|\psi_{n+k}(\omega) - \psi_n(\omega) - \psi_k(\sigma^n \omega)| \leq C(V, \alpha) =: C(\Psi).$$

Thus Ψ is almost additive.

Recall that $N \geq 4 + \hat{n}$. By (2.4) we have

$$(4.4) \quad (n + N) \ln c_3 \leq \psi_n(\omega) \leq -n \ln 2.$$

Thus $\Psi \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$.

By the definition, if $u \in \Omega_n^{(\kappa)}$ and $\omega, \tilde{\omega} \in [u]$, then

$$\psi_n(\omega) = \ln |B_{w^{u_0}u[1, \dots, n]}| = \psi_n(\tilde{\omega}).$$

So Ψ has bounded variation property with constant $D(\Psi) = 0$. \square

4.3. Weak-Gibbs metric on $\Omega^{(\kappa)}$.

Since $\Psi \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$, we can define the weak-Gibbs metric d_Ψ on $\Omega^{(\kappa)}$ according to (2.6) as follows. Given $\omega, \tilde{\omega} \in \Omega^{(\kappa)}$. If $\omega_0 \neq \tilde{\omega}_0$, define $d_\Psi(\omega, \tilde{\omega}) := \text{diam}(\Sigma_{\alpha, V})$. If $\omega_0 = \tilde{\omega}_0$ define

$$d_\Psi(\omega, \tilde{\omega}) := \sup_{\omega' \in [\omega \wedge \tilde{\omega}]} \exp(\psi_{|\omega \wedge \tilde{\omega}|}(\omega')) = |B_{w^{\omega_0} \star (\omega \wedge \tilde{\omega})}|.$$

Denote by $|\cdot|$ the standard metric on \mathbb{R} . Then we have

Proposition 1. $\pi_{\tilde{w}} : (\Omega^{(\kappa)}, d_\Psi) \rightarrow (\Sigma_{\tilde{w}}, |\cdot|)$ is a bi-Lipschitz homeomorphism.

Proof. Given $\omega, \tilde{\omega} \in \Omega^{(\kappa)}$. Assume $\omega|_n = \tilde{\omega}|_n$ and $\omega_{n+1} \neq \tilde{\omega}_{n+1}$. Then we have $d_\Psi(\omega, \tilde{\omega}) = |B_{w^{\omega_0} \star \omega|_n}|$. Write $x := \pi_{\tilde{w}}(\omega)$ and $y := \pi_{\tilde{w}}(\tilde{\omega})$. It is seen that $x, y \in B_{w^{\omega_0} \star \omega|_n}$, consequently

$$|x - y| \leq |B_{w^{\omega_0} \star \omega|_n}| = d_\Psi(\omega, \tilde{\omega}).$$

On the other hand, since $\omega_{n+1} \neq \tilde{\omega}_{n+1}$, there is a gap G of order $n + N$ which is contained in (x, y) . By Lemma 6, there exists a constant $C = C(\alpha, V)$ such that

$$|x - y| \geq |G| \geq C|B_G| = C|B_{w^{\omega_0} \star \omega|_n}| = Cd_\Psi(\omega, \tilde{\omega}).$$

Thus $\pi_{\tilde{w}}$ is a bi-Lipschitz homeomorphism. \square

Remark 2. This proposition is crucial for studying the dimensional properties of the spectrum and the density of states measure. Because by this proposition, the metric property on $(\Omega^{(\kappa)}, d_\Psi)$ is the same with that on $(\Sigma_{\tilde{w}}, |\cdot|)$ (see for example [17] chapter 2). Thus we can convert the dimension problem of the spectrum completely to that of the symbolic space. We will use this proposition repeatedly in what follows.

4.4. Bowen's formula, Hausdorff dimension and Hausdorff measure of $\Sigma_{\tilde{w}}$.

Since $\Psi \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$, by Corollary 1, $P(s\Psi) = 0$ has a unique solution $\tilde{s}_V = \tilde{s}_{V, \tilde{w}}$. By Lemma 7, Ψ has bounded variation, so does $\tilde{s}_V \Psi$. Let m be the unique Gibbs measure related to $\tilde{s}_V \Psi$. Then

Theorem 8. $m \asymp \mathcal{H}^{\tilde{s}_V}|_{\Omega^{(\kappa)}}$. Thus $0 < \mathcal{H}^{\tilde{s}_V}(\Omega^{(\kappa)}) < \infty$. Consequently $\dim_H m = \dim_H \Omega^{(\kappa)} = \tilde{s}_V$ and $\mathcal{H}^{\tilde{s}_V}|_{\Omega^{(\kappa)}}$ is exact dimensional. Moreover

$$(4.5) \quad \tilde{s}_V = \frac{h_m(\sigma)}{-\Psi_*(m)}.$$

Proof. By Theorem 5 (i), there exists a constant $C > 1$ such that for any $u \in \Omega_n^{(\kappa)}$

$$C^{-1} \leq \frac{m([u])}{\exp(\tilde{s}_V \psi_n(\omega))} \leq C \quad (\forall \omega \in [u]).$$

Together with the definition of weak-Gibbs metric, we conclude that

$$(4.6) \quad C^{-1} \text{diam}([u])^{\tilde{s}_V} \leq m([u]) \leq C \text{diam}([u])^{\tilde{s}_V}.$$

Then by the definition of Hausdorff measure it is ready to show that $\mathcal{H}^{\tilde{s}_V}|_{\Omega^{(\kappa)}} \asymp m$. Consequently $0 < \mathcal{H}^{\tilde{s}_V}(\Omega^{(\kappa)}) < \infty$ and $\dim_H \Omega^{(\kappa)} = \tilde{s}_V$.

On the other hand (4.6) obviously implies that $d_m(x) = \tilde{s}_V$ for all $\omega \in \Omega$. Thus m and $\mathcal{H}^{\tilde{s}_V}|_{\Omega^{(\kappa)}}$ are exact dimensional. Finally by (2.7) we get

$$\tilde{s}_V = \dim_H m = \frac{h_m(\sigma)}{-\Psi_*(m)}.$$

□

Remark 3. The equality $\tilde{s}_V = \dim_H \Omega^{(\kappa)}$ is known as Bowen's formula and (4.5) is kind of Young's formula. Both are very famous in the classical theories. Since $\dim_H m$ reach the dimension of the whole space $\Omega^{(\kappa)}$, we say that m is the measure of maximal dimension.

Now we introduce the notion of "Gibbs type measure".

Definition 3. Assume X is a compact metric space and can be written as

$$X = \bigcup_{i=1}^m \bigcup_{j=1}^{n_i} X_{ij}$$

where X_{ij} are compact and disjoint. μ is a measure supported on X . We say that μ is a *Gibbs type measure*, if there exists a topologically mixing subshift of finite type (Σ_A, σ) with alphabet $\mathcal{A} = \{1, \dots, m\}$ such that for any choice $\tau = (\tau_1, \dots, \tau_m)$, $1 \leq \tau_i \leq n_i$; $1 \leq i \leq m$, there exist a weak Gibbs metric d_τ on Σ_A , a Gibbs measure ν_τ on Σ_A and a bi-Lipschitz map $\phi^\tau : \Sigma_A \rightarrow X_\tau = \bigcup_{i=1}^m X_{i\tau_i}$ such that

$$\phi^\tau([i]) = X_{i\tau_i} \quad \text{and} \quad \phi_*^\tau(\nu_\tau) \asymp \mu|_{X_\tau}.$$

Lemma 8. If X is defined as above and μ is a Gibbs type measure supported on X , then μ is exact dimensional.

Proof. By Theorem 7, for each τ , ν_τ is exact dimensional. Since ϕ^τ is bi-Lipschitz, we conclude that $\mu|_{X_\tau}$ is exact dimensional. Especially $\mu|_{X_{i\tau_i}}$ is exact dimensional and the dimensions of $\mu|_{X_{i\tau_i}}$ are the same for $1 \leq i \leq m$. Since we can choose τ freely, we conclude that $\mu = \mu|_X$ is exact dimensional. □

By using Proposition 1 we have the following consequence.

Theorem 9. $\mathcal{H}^{\tilde{s}_V}|_{\Sigma_{\vec{w}}}$ is a Gibbs type measure, consequently $\mathcal{H}^{\tilde{s}_V}|_{\Sigma_{\vec{w}}}$ is exact dimensional and $0 < \mathcal{H}^{\tilde{s}_V}(\Sigma_{\vec{w}}) < \infty$. Thus $\dim_H \Sigma_{\vec{w}} = \tilde{s}_V$ and

$$(4.7) \quad \tilde{s}_V = \frac{h_m(\sigma)}{-\Psi_*(m)}.$$

Proof. By Proposition 1, $\pi_{\vec{w}}$ is a bi-Lipschitz homeomorphism. By general principle on Hausdorff measure (see for example [17]) We conclude that $(\pi_{\vec{w}})_*(\mathcal{H}^{\vec{s}_V}|_{\Omega(\kappa)}) \asymp \mathcal{H}^{\vec{s}_V}|_{\Sigma_{\vec{w}}}$. By Theorem 8, we get $(\pi_{\vec{w}})_*(m) \asymp \mathcal{H}^{\vec{s}_V}|_{\Sigma_{\vec{w}}}$, thus $\mathcal{H}^{\vec{s}_V}|_{\Sigma_{\vec{w}}}$ is a Gibbs type measure. By Lemma 8, it is exact dimensional. The other results follow from Theorem 8 and the fact that $\pi_{\vec{w}}$ is bi-Lipschitz. \square

5. THE DENSITY OF STATES MEASURE

In this section we study $\mathcal{N}_{\alpha,V}$. We will show that $\mathcal{N}_{\alpha,V}$ is a Markov measure and in some sense the measure of maximal entropy. Recall that $\alpha = [a_0; a_1, \dots, a_{\hat{n}}; \kappa, \kappa, \dots]$.

5.1. $\mathcal{N}_{\alpha,V}$ is a Markov measure.

Let H_n be the restriction of $H_{\alpha,V,0}$ to the box $[1, q_n]$ with periodic boundary condition. Let $\mathcal{X}_n = \{x_{n,1}, \dots, x_{n,q_n}\}$ be the eigenvalues of H_n . Recall that σ_n is defined in (2.2).

Lemma 9. *Each band in σ_n contains exactly one value in \mathcal{X}_n .*

Proof. This comes from the Bloch theory. Write $u^n = (v_1 \cdots v_{q_n})^{\mathbb{Z}}$ and define $H^{(n)} = H_{u^n}$, then $\sigma_n = \sigma(H_{u^n})$ is made of q_n disjoint bands. There is another way to represent the spectrum by using the Bloch wave. Consider the solution of $H^{(n)}\psi_\theta = x\psi_\theta$ of the Bloch type, i.e. $\psi_\theta(m) = e^{im\theta}u(m)$ with $u(m) = u(m+q_n)$ and some $\theta \in [0, 2\pi]$. When $\theta \in [0, 2\pi]$ is fixed, $H^{(n)}\psi_\theta = x\psi_\theta$ has exact q_n solutions. We denote the set of eigenvalues as E_θ . Then each band in σ_n contains exact one point of E_θ , moreover

$$\sigma_n = \bigcup_{\theta \in [0, 2\pi]} E_\theta.$$

Now it is direct to check that $\mathcal{X}_n = E_0$ is the set of endpoint of each band which is related to the phase $\theta = 0$. \square

Define

$$\nu_n = \frac{1}{q_n} \sum_{i=1}^{q_n} \delta_{x_{n,i}}.$$

It is well known that $\nu_n \rightarrow \mathcal{N}_{\alpha,V}$ weakly (see for example [8]).

Lemma 10. *There exist constants $C_\alpha > 0$ and*

$$(5.1) \quad C_I = \frac{\alpha_\kappa}{1 + \alpha_\kappa^2}, \quad C_{II} = \frac{\alpha_\kappa^2}{1 + \alpha_\kappa^2} \quad \text{and} \quad C_{III} = \frac{\alpha_\kappa(\alpha_\kappa - 1)}{1 + \alpha_\kappa^2}$$

such that for any $n > \hat{n}$ and $B_w \in \mathcal{B}_n$ we have

$$\mathcal{N}_{\alpha,V}(B_w) = C_\alpha C_{t_w} \alpha_\kappa^{-n}.$$

Proof. By Lemma 1, \hat{A}_κ has eigenvalues $\alpha_\kappa, -1$ and $-1/\alpha_\kappa$, then there exists invertible matrix P such that $\hat{A}_\kappa = P \cdot \text{diag}(\alpha_\kappa, -1, -1/\alpha_\kappa) \cdot P^{-1}$. By (2.1), for $l \geq \hat{n}$ we have

$$\begin{pmatrix} q_l \\ q_{l-1} \end{pmatrix} = \begin{pmatrix} \kappa & 1 \\ 1 & 0 \end{pmatrix}^{l-\hat{n}} \begin{pmatrix} a_{\hat{n}} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From this it is easy to show that there exist two constants $c_\alpha > 0, d_\alpha$ such that

$$(5.2) \quad q_l = c_\alpha \alpha_\kappa^l + d_\alpha (-\alpha_\kappa)^{-l}.$$

Notice that $a_k = \kappa$ for any $k \geq n$ since $n > \hat{n}$. Thus for any $l > n$, by Lemma 2 and Lemma 9, we have

$$\begin{aligned} \nu_l(B_w) &= \sum_{|u|=l+1, w \prec u} \nu_l(B_u) \\ &= \frac{\#\{u : |u| = l+1, w \prec u, t_u = II, III\}}{q_l} \\ &= \frac{v^{tw} \cdot \hat{A}_\kappa^{l-n} \cdot v_*}{c_\alpha \alpha_\kappa^l + d_\alpha (-\alpha_\kappa)^{-l}} \quad (\text{by (5.2) and (2.3)}) \\ &= \frac{v^{tw} \cdot P \cdot \text{diag}(\alpha_\kappa^{l-n}, (-1)^{l-n}, (-\alpha_\kappa)^{n-l}) \cdot P^{-1} \cdot v_*}{c_\alpha \alpha_\kappa^l + d_\alpha (-\alpha_\kappa)^{-l}} \\ &= \frac{C_{tw} \alpha_\kappa^{l-n} + C_{tw,2} (-1)^{l-n} + C_{tw,3} (-\alpha_\kappa)^{n-l}}{c_\alpha \alpha_\kappa^l + d_\alpha (-\alpha_\kappa)^{-l}}. \end{aligned}$$

Since $B_w \cap \Sigma_{\alpha,V}$ is open and closed in $\Sigma_{\alpha,V}$ and $\nu_l \rightarrow \mathcal{N}_{\alpha,V}$ weakly, by taking a limit we get

$$\mathcal{N}_{\alpha,V}(B_w) = \frac{C_{tw}}{c_\alpha} \alpha_\kappa^{-n} =: C_\alpha C_{tw} \alpha_\kappa^{-n}.$$

By a simple computation, we get (5.1). \square

Recall that we denote $e_j := e_{\kappa,j}$ for simplicity. Define a matrix $Q = (q_{e_i e_j})$ of order $2\kappa + 2$ as

$$(5.3) \quad q_{e_i e_j} = \begin{cases} \frac{C_{te_j}}{C_{te_i}} \cdot \alpha_\kappa^{-1} & e_i \rightarrow e_j \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2. *Q is a primitive stochastic matrix. Let $p = (p_{e_1}, \dots, p_{e_{2\kappa+2}})$ be the stationary distribution of Q , i.e. the unique probability vector p such that $pQ = p$, then*

$$(5.4) \quad p_{e_{\kappa+2}} = \frac{\alpha_\kappa}{\kappa \alpha_\kappa + 2}.$$

Proof. By the definition, $q_{e_i e_j} > 0$ if and only if $A_\kappa(i, j) > 0$. Since A_κ is primitive, so is Q . Denote the i -th row of Q by q^i . By the definition of $q_{e_i e_j}$ and (5.1), we have

$$(5.5) \quad q^i = \begin{cases} (0, \dots, 0, 1, 0, \dots, 0) & i = 1, \dots, \kappa + 1 \\ \left(\underbrace{\frac{1}{\alpha_\kappa^2}, \dots, \frac{1}{\alpha_\kappa^2}}_{\kappa+1}, 0, \underbrace{\frac{\alpha_\kappa - 1}{\alpha_\kappa^2}, \dots, \frac{\alpha_\kappa - 1}{\alpha_\kappa^2}}_{\kappa} \right) & i = \kappa + 2 \\ \left(\underbrace{\frac{1}{\alpha_\kappa(\alpha_\kappa - 1)}, \dots, \frac{1}{\alpha_\kappa(\alpha_\kappa - 1)}}_{\kappa}, 0, 0, \underbrace{\frac{1}{\alpha_\kappa}, \dots, \frac{1}{\alpha_\kappa}}_{\kappa-1}, 0 \right) & i = \kappa + 3, \dots, 2\kappa + 2. \end{cases}$$

It is seen that Q is a stochastic matrix.

We write $p_i = p_{e_i}$ temporarily. Write $\delta := \alpha_\kappa^{-2}$ and $\beta := (\alpha_\kappa(\alpha_\kappa - 1))^{-1}$. Since $p = pQ$, we have

$$\begin{cases} \delta p_{\kappa+2} + \beta(p_{\kappa+3} + \dots + p_{2\kappa+2}) & = p_i \quad (i = 1 \dots, \kappa) \\ \delta p_{\kappa+2} & = p_{\kappa+1} \\ p_1 + \dots + p_{\kappa+1} & = p_{\kappa+2}. \end{cases}$$

Then we have

$$p_{\kappa+2} = p_1 + \dots + p_{\kappa+1} = (\kappa + 1)\delta p_{\kappa+2} + \kappa\beta(p_{\kappa+3} + \dots + p_{2\kappa+2}).$$

Notice that p is a probability vector, thus we have

$$1 = (p_1 + \dots + p_{\kappa+1}) + p_{\kappa+2} + (p_{\kappa+3} + \dots + p_{2\kappa+2}) = (2 + \frac{(\alpha_\kappa - 1)^2}{\alpha_\kappa})p_{\kappa+2}.$$

Then we get $p_{\kappa+2} = \alpha_\kappa / (\kappa\alpha_\kappa + 2)$. □

Now we have the following structure for $\mathcal{N}_{\alpha, V}$:

Proposition 3. *For any $n > \hat{n}$ and any $u \in \Omega_n^{(\alpha)}$, the measure $\mathcal{N}_{\alpha, V}|_{B_u}$ is a Markov measure with transition matrix Q .*

Proof. For any $uw_1 \dots w_k \in \Omega_{n+k}^{(\alpha)}$, by using Lemma 10 repeatedly, we have

$$\begin{aligned} \mathcal{N}_{\alpha, V}(X_{uw_1 \dots w_k}) &= \mathcal{N}_{\alpha, V}(X_u) \cdot \frac{\mathcal{N}_{\alpha, V}(X_{uw_1})}{\mathcal{N}_{\alpha, V}(X_u)} \dots \frac{\mathcal{N}_{\alpha, V}(X_{uw_1 \dots w_k})}{\mathcal{N}_{\alpha, V}(X_{uw_1 \dots w_{k-1}})} \\ &= \mathcal{N}_{\alpha, V}(X_u) q_{w_0 w_1} q_{w_1 w_2} \dots q_{w_{k-1} w_k}, \end{aligned}$$

where w_0 is the last letter of u . Then the result follows. □

5.2. $\mathcal{N}_{\alpha, V}$ and the measure of maximal entropy.

To study the dimension property of $\mathcal{N}_{\alpha, V}$, we need to go back to the TDS $(\Omega^{(\kappa)}, \sigma)$ again. We will establish the relation between $\mathcal{N}_{\alpha, V}$ and the measure of maximal entropy and obtain the dimension formula of $\mathcal{N}_{\alpha, V}$.

Define stochastic matrix Q according to (5.3). Let μ_Q be the unique invariant Markov measure on the subshift $(\Omega^{(\kappa)}, \sigma)$ with transition matrix Q . It is well known that μ_Q is a Gibbs measure on $\Omega^{(\kappa)}$ with additive potential $\phi : \Omega^{(\kappa)} \rightarrow \mathbb{R}$ defined by

$$\phi(\omega) := \ln q_{\omega_0 \omega_1}.$$

Note that $\phi \leq 0$. Write $\Phi = (\phi_n)_{n=0}^\infty$ with $\phi_n = S_n \phi$, then Φ is a family of additive potentials. By the definition of Gibbs measure it is not hard to compute that $P(\Phi) = 0$.

Theorem 10. μ_Q is the measure of maximal entropy and the Parry measure of the subshift $(\Omega^{(\kappa)}, \sigma)$. Moreover μ_Q is exact dimensional with

$$(5.6) \quad \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_*(\mu_Q)} = \frac{\ln \alpha_\kappa}{-\Psi_*(\mu_Q)}.$$

Proof. At first since $(\Omega^{(\kappa)}, \sigma)$ is a topologically mixing subshift and the incidence matrix A_κ has Perron-Frobenius eigenvalue α_κ , we have $h_{top}(\sigma) = \ln \alpha_\kappa$ (see for example [41]).

Recall that p is the stationary distribution satisfying $pQ = p$. Then for any $u \in \Omega_n^{(\kappa)}$, we have $\mu_Q([u]) = p_{u_0} q_{u_0 u_1} \cdots q_{u_{n-1} u_n}$. For each $e \in \mathcal{A}_\kappa$, fix some $w^e \in \Omega_N^{(\alpha)}$ such that $w_N^e = e$. Then by Proposition 3,

$$\mu_Q([u]) = p_{u_0} q_{u_0 u_1} \cdots q_{u_{n-1} u_n} = \frac{p_{u_0}}{\mathcal{N}_{\alpha, V}(X_{w^{u_0} u}[1, \dots, n])} \mathcal{N}_{\alpha, V}(X_{w^{u_0} u}[1, \dots, n]).$$

Then by Lemma 10, for any $\omega \in \Omega^{(\kappa)}$,

$$\lim_{n \rightarrow \infty} \frac{-\ln \mu_Q([\omega|_n])}{n} = \lim_{n \rightarrow \infty} \frac{-\ln \mathcal{N}_{\alpha, V}(X_{w^{\omega_0} \omega}[1, \dots, n])}{n} = \ln \alpha_\kappa = h_{top}(\sigma).$$

By Shannon-McMillan-Breiman Theorem, we conclude that $h_{\mu_Q}(\sigma) = h_{top}(\sigma)$. Since the measure of maximal entropy is unique, which is the so called Parry measure (see for example [41] chapter 8), we conclude that μ_Q is the the measure of maximal entropy and the Parry measure of the system $(\Omega^{(\kappa)}, \sigma)$.

Since μ_Q is a Gibbs measure, by Theorem 7, μ_Q is exact dimensional and the Hausdorff dimension of μ_Q is given by (5.6). \square

Remark 4. The proof indeed gives that for any $u \in \Omega_n^{(\kappa)}$

$$(5.7) \quad \mu_Q([u]) \sim \alpha_\kappa^{-n}.$$

We fix again a $\vec{w} \in \mathcal{D}$ and let $\pi_{\vec{w}}$ be defined as in (4.2).

Theorem 11. $\mathcal{N}_{\alpha, V}|_{\Sigma_{\vec{w}}}$ is a Gibbs type measure. It is exact dimensional and

$$(5.8) \quad \dim_H \mathcal{N}_{\alpha, V}|_{\Sigma_{\vec{w}}} = \frac{\ln \alpha_\kappa}{-\Psi_*(\mu_Q)}.$$

Proof. Write $\nu_Q := (\pi_{\vec{w}})_*(\mu_Q)$. By Proposition 3 we have

$$\nu_Q(X_{w^{u_0}u[1, \dots, n]}) = \mu_Q([u]) = p_{u_0}q_{u_0u_1} \cdots q_{u_{n-1}u_n} = \frac{p_{u_0}}{\mathcal{N}_{\alpha,V}(X_{w^{u_0}})} \mathcal{N}_{\alpha,V}(X_{w^{u_0}u[1, \dots, n]}).$$

Consequently $\nu_Q \asymp \mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$. Since μ_Q is a Gibbs measure, we conclude that $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$ is a Gibbs type measure. Since $\pi_{\vec{w}} : \Omega^{(\kappa)} \rightarrow \Sigma_{\vec{w}}$ is bi-Lipschitz and $\nu_Q \asymp \mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$, we conclude that both measures ν_Q and $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$ are exact dimensional and has the same Hausdorff dimension with μ_Q . \square

6. MULTIFRACTAL ANALYSIS AND OPTIMAL HÖLDER EXPONENT OF $\mathcal{N}_{\alpha,V}$

In this section we study the optimal Hölder exponent of $\mathcal{N}_{\alpha,V}$ restricted to the dynamical subset $\Sigma_{\vec{w}}$. We will see that the exponent can be obtained from the multifractal analysis of $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$.

At first we conduct the multifractal analysis of μ_Q , then by the bi-Lipschitz homeomorphism $\pi_{\vec{w}}$, the result is converted to that of $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$.

6.1. Multifractal analysis of μ_Q .

We begin with two useful lemmas:

Lemma 11. *For any $\omega \in \Omega^{(\kappa)}$, we have*

$$(6.1) \quad \underline{d}_{\mu_Q}(\omega) = \liminf_{n \rightarrow \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} \quad \text{and} \quad \bar{d}_{\mu_Q}(\omega) = \limsup_{n \rightarrow \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)}.$$

Proof. Fix $\omega \in \Omega^{(\kappa)}$ and $r > 0$ very small. Assume n is the unique number such that

$$(6.2) \quad \exp(\psi_{nn_0}(\omega)) = |B_{w^{\omega_0}\omega[1, \dots, nn_0]}| < r \leq |B_{w^{\omega_0}\omega[1, \dots, (n-1)n_0]}| = \exp(\psi_{(n-1)n_0}(\omega)),$$

where n_0 is given in Lemma 4. Thus $[\omega|_{nn_0}] \subset B(\omega, r) \subset [\omega|_{(n-1)n_0}]$. Consequently

$$(6.3) \quad \mu_Q([\omega|_{nn_0}]) \leq \mu_Q(B(\omega, r)) \leq \mu_Q([\omega|_{(n-1)n_0}]).$$

Notice that for any $u \in \Omega_n^{(\kappa)}$ and $\omega' \in [u]$, we have

$$\mu_Q([u]) = p_{u_0}q_{u_0u_1} \cdots q_{u_{n-1}u_n} = p_{u_0} \exp(S_n \phi(\omega')) = p_{u_0} \exp(\phi_n(\omega')).$$

Combine (6.2) and (6.3) we get

$$\frac{\phi_{(n-1)n_0}(\omega) + \ln p_{\omega_0}}{\psi_{nn_0}(\omega)} \leq \frac{\ln \mu_Q(B(\omega, r))}{\ln r} \leq \frac{\phi_{nn_0}(\omega) + \ln p_{\omega_0}}{\psi_{(n-1)n_0}(\omega)}.$$

By (2.4), $c_3^{N+nn_0} \leq r \leq 2^{2-N-(n-1)n_0}$, thus $n \rightarrow \infty$ when $r \rightarrow 0$. By (5.7), $\phi_n(\omega) \leq -n \ln \alpha_\kappa + c$. By (4.4), $\psi_n(\omega) \leq -n \ln 2$. Now by using the definition of almost additive potential and taking the upper and lower limits we get the result. \square

Lemma 12. *There exists $d_1 = d_1(\kappa) < d_2 = d_2(\kappa) < 0$ such that for any $\mu \in \mathcal{M}(\Omega^{(\kappa)}, \sigma)$, we have*

$$(6.4) \quad d_1 \leq \Phi_*(\mu) \leq d_2.$$

Proof. Since Φ is additive and μ is invariant, we have

$$\Phi_*(\mu) = \int_{\Omega^{(\kappa)}} \phi d\mu = \int_{\Omega^{(\kappa)}} \frac{S_3\phi}{3} d\mu.$$

By the definition we get

$$S_3\phi(\omega) = \ln q_{\omega_0\omega_1} q_{\omega_1\omega_2} q_{\omega_2\omega_3}.$$

We discuss two cases. At first we assume $\kappa = 1$. By (5.5) we know that

$$q_{e_i e_j} \begin{cases} = 1 & (i, j) = (1, 3); (2, 3); (4, 1) \\ \in (0, 1) & (i, j) = (3, 1); (3, 2); (3, 4) \\ = 0 & \text{otherwise} \end{cases}$$

It is seen that if $e_{i_0}e_{i_1}e_{i_2}$ is admissible, then there exists at least one $j \in \{0, 1, 2\}$ such that $i_j = 3$. Write $q_{\min} = \min\{q_{e_3e_1}, q_{e_3e_2}, q_{e_3e_4}\}$ and $q_{\max} = \max\{q_{e_3e_1}, q_{e_3e_2}, q_{e_3e_4}\}$ then $0 < q_{\min} \leq q_{\max} < 1$. Thus

$$3 \ln q_{\min} \leq S_3\phi(\omega) \leq \ln q_{\max}.$$

Next we assume $\kappa \geq 2$. By (5.5), $q_{e_i e_j} = 1$ if and only if $i \leq \kappa + 1$ and $j = \kappa + 2$. Write $q_{\min} := \min\{q_{e_i e_j} : e_i \rightarrow e_j; q_{e_i e_j} \neq 1\}$ and $q_{\max} := \max\{q_{e_i e_j} : e_i \rightarrow e_j; q_{e_i e_j} \neq 1\}$. Then $0 < q_{\min} \leq q_{\max} < 1$. From the structure of A_κ , it is ready to see that if $e_{i_0}e_{i_1}e_{i_2}$ is admissible, then $i_1 \neq \kappa + 2$ or $i_2 \neq \kappa + 2$. Thus we still have

$$3 \ln q_{\min} \leq S_3\phi(\omega) \leq \ln q_{\max}.$$

Take $d_1 = \ln q_{\min}$ and $d_2 = \ln q_{\max}/3$ we get the result. \square

Consider the function $Q(q, t) := P(q\Phi + t\Psi)$. By Corollary 1, since $\Psi \in C_{aa}^-(\Omega^{(\kappa)}, \sigma)$, for each $q \in \mathbb{R}$ fixed, there exists a unique number $\tau(q)$ such that $Q(q, \tau(q)) = 0$. Define

$$\mathcal{B} := \left\{ \frac{\Phi_*(\mu)}{\Psi_*(\mu)} : \mu \in \mathcal{M}(\Omega^{(\kappa)}, \sigma) \right\}.$$

Then $\mathcal{B} = [\beta_*, \beta^*]$ is an interval.

Theorem 12. (i) Define $\Lambda_\beta := \{\omega \in \Omega^{(\kappa)} : d_{\mu_Q}(\omega) = \beta\}$. Then $\Lambda_\beta \neq \emptyset$ if and only if $\beta \in \mathcal{B}$. For any $\beta \in \mathcal{B}$

$$\dim_H \Lambda_\beta = \tau^*(\beta) := \inf_{q \in \mathbb{R}} (\tau(q) + \beta q).$$

(ii) There exist two constants $0 < C_1(\alpha, V) \leq C_2(\alpha, V)$ such that $C_1 \leq \beta_* \leq \beta^* \leq C_2$. Moreover

$$(6.5) \quad \beta_* = \inf_{\omega \in \Omega} \underline{d}_{\mu_Q}(\omega) \quad \text{and} \quad \beta^* = \sup_{\omega \in \Omega} \overline{d}_{\mu_Q}(\omega).$$

Thus β_* is the optimal Hölder exponent of μ_Q .

Proof. (i) Recall the definition in Section 2.4.2. By (6.1) we have $\Lambda_\beta = \Lambda_{\Phi/\Psi}(\beta)$ and $\tau(q) = \mathcal{L}_{\Phi/\Psi}(q, \beta) - q\beta$. Thus by Theorem 6 we have $\Lambda_\beta \neq \emptyset$ if and only if $\beta \in \mathcal{B}$. Moreover if $\beta \in \mathcal{B}$ we have

$$\dim_H \Lambda_\beta = \mathcal{L}_{\Phi/\Psi}^*(\beta) = \inf_{q \in \mathbb{R}} (\tau(q) + \beta q).$$

(ii) By (4.4), for any invariant measure μ we have

$$\ln c_3 \leq \Psi_*(\mu) \leq -\ln 2 < 0.$$

By (6.4), for any invariant measure μ we have

$$d_1 \leq \Phi_*(\mu) \leq d_2 < 0.$$

From this we conclude that

$$0 < C_1 := \frac{d_2}{\ln c_3} \leq \beta_* \leq \frac{\Phi_*(\mu)}{\Psi_*(\mu)} \leq \beta^* \leq \frac{d_1}{-\ln 2} =: C_2.$$

Now fix any $\omega \in \Omega^{(\kappa)}$. By Lemma 11 we have

$$\underline{d}_{\mu_Q}(\omega) = \liminf_{n \rightarrow \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} = \lim_{k \rightarrow \infty} \frac{\phi_{n_k}(\omega)}{\psi_{n_k}(\omega)}.$$

By choosing a further subsequence we can further assume that $(\sum_{j=0}^{n_k-1} \delta_{\sigma^j \omega})/n_k \rightarrow \mu$. Then μ is invariant and by [20] Lemma A.4(ii), we have

$$\lim_{k \rightarrow \infty} \frac{\phi_{n_k}(\omega)}{n_k} = \Phi_*(\mu) \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\psi_{n_k}(\omega)}{n_k} = \Psi_*(\mu).$$

Thus $\underline{d}_{\mu_Q}(\omega) \in \mathcal{B}$. Similarly we can show that $\bar{d}_{\mu_Q}(\omega) \in \mathcal{B}$. On the other hand, since Λ_{β_*} and Λ_{β^*} are all nonempty, there exist ω_* and ω^* such that $d_{\mu_Q}(\omega_*) = \beta_*$ and $d_{\mu_Q}(\omega^*) = \beta^*$. Thus (6.5) holds. \square

In the following we will give another expression for the optimal Hölder exponent of μ_Q , which is more convenient when we study the asymptotic property.

Write $\psi_{n,\min} := \inf\{\psi_n(\omega) : \omega \in \Omega^{(\kappa)}\}$. By the almost additivity of Ψ we have

$$\psi_{n+k,\min} \geq \psi_{n,\min} + \psi_{k,\min} - C(\Psi).$$

Thus $\{C(\Psi) - \psi_{n,\min} : n \geq 0\}$ form a sub-additive sequence. Then it is well known that the following limit exists

$$\lim_{n \rightarrow \infty} \frac{\psi_{n,\min}}{n} = \sup_{n \geq 0} \frac{\psi_{n,\min} - C(\Psi)}{n} =: \Psi_{\min}.$$

Notice that by (4.4), we have $\Psi_{\min} \leq -\ln 2 < 0$.

Proposition 4. *We have*

$$\gamma_{\mu_Q} = \frac{\ln \alpha_\kappa}{-\Psi_{\min}}.$$

Proof. At first we show $\gamma_{\mu_Q} \geq -\ln \alpha_\kappa / \Psi_{\min}$. It is sufficient to show that $\underline{d}_{\mu_Q}(\omega) \geq -\ln \alpha_\kappa / \Psi_{\min}$ for any $\omega \in \Omega^{(\kappa)}$. By the definition of ϕ_n and μ_Q we have

$$\phi_n(\omega) = \ln q_{\omega_0 \omega_1} \cdots q_{\omega_{n-1} \omega_n} = \ln \frac{\mu_Q([\omega|_n])}{p_{\omega_0} q_{\omega_n \omega_{n+1}}}.$$

By (5.7) we have

$$(6.6) \quad \lim_{n \rightarrow \infty} \frac{\phi_n(\omega)}{n} = \lim_{n \rightarrow \infty} \frac{\ln \mu_Q([\omega|_n])}{n} = -\ln \alpha_\kappa.$$

On the other hand we have

$$(6.7) \quad \liminf_{n \rightarrow \infty} \frac{\psi_n(\omega)}{n} \geq \lim_{n \rightarrow \infty} \frac{\psi_{n,\min}}{n} = \Psi_{\min}.$$

Combining (6.6), (6.7) and (6.1) we get

$$\underline{d}_{\mu_Q}(\omega) = \liminf_{n \rightarrow \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} \geq \frac{\ln \alpha_\kappa}{-\Psi_{\min}}.$$

Next we show $\gamma_{\mu_Q} \leq -\ln \alpha_\kappa / \Psi_{\min}$. It is sufficient to show that for any $\epsilon > 0$ small $\underline{d}_{\mu_Q}(\omega^\epsilon) < \ln \alpha_\kappa / (-\Psi_{\min} - \epsilon)$ for some $\omega^\epsilon \in \Omega^{(\kappa)}$. By (6.6), it is sufficient to show that $\liminf_{n \rightarrow \infty} \psi_n(\omega^\epsilon)/n \leq \Psi_{\min} + \epsilon$ for some $\omega^\epsilon \in \Omega^{(\kappa)}$. We find such a ω^ϵ as follows. Recall that the incidence matrix A_κ is primitive, thus there exists N_κ such that $A_\kappa^{N_\kappa-2}$ are positive. At first take n_0 big enough such that

$$(6.8) \quad \frac{\|\psi_{N_\kappa}\|_\infty}{n_0} \leq \frac{\epsilon}{8}, \quad \frac{C(\Psi)}{n_0} \leq \frac{\epsilon}{8}, \quad -4N_\kappa \Psi_{\min} \leq n_0 \epsilon \quad \text{and} \quad \psi_{n_0,\min} \leq n_0(\Psi_{\min} + \epsilon/4).$$

Let $\tilde{\omega} \in \Omega^{(\kappa)}$ such that $\psi_{n_0}(\tilde{\omega}) = \psi_{n_0,\min}$. Since $A_\kappa^{N_\kappa-2}$ is positive, we can find $w \in \Omega_{N_\kappa-2}^{(\kappa)}$ such that both $u := \tilde{\omega}|_{n_0} w$ and $w\tilde{\omega}|_{n_0}$ are admissible. Thus $\omega^\epsilon := u^\infty \in \Omega^{(\kappa)}$. Notice that $|u| = |\tilde{\omega}|_{n_0}|w| = n_0 + 1 + N_\kappa - 1 = n_0 + N_\kappa$. Let $n_1 = n_0 + N_\kappa$. By the definition of Ψ we have $\psi_{n_0}(\tilde{\omega}) = \psi_{n_0}(\omega^\epsilon)$. By almost additivity and (6.8) we have

$$(6.9) \quad \begin{aligned} \psi_{n_1}(\omega^\epsilon) &\leq \psi_{n_0}(\omega^\epsilon) + \psi_{N_\kappa}(\sigma^{n_0} \omega^\epsilon) + C(\Psi) \leq \psi_{n_0}(\tilde{\omega}) + \frac{n_0 \epsilon}{8} + \frac{n_0 \epsilon}{8} \\ &\leq n_0(\Psi_{\min} + \frac{\epsilon}{4}) + \frac{n_0 \epsilon}{4} \leq n_1(\Psi_{\min} + 3\epsilon/4). \end{aligned}$$

Notice that by the definition of ω^ϵ we have $\sigma^{jn_1} \omega^\epsilon = \omega^\epsilon$ for any $j \geq 0$. Again by almost additivity we get

$$\psi_{kn_1}(\omega^\epsilon) \leq \sum_{j=0}^{k-1} \psi_{n_1}(\sigma^{jn_1} \omega^\epsilon) + (k-1)C(\Psi) \leq k\psi_{n_1}(\omega^\epsilon) + kC(\Psi).$$

Combining with (6.9) and (6.8) we conclude that

$$\psi_{kn_1}(\omega^\epsilon) \leq kn_1(\Psi_{\min} + \epsilon).$$

Consequently we have

$$\liminf_{n \rightarrow \infty} \frac{\psi_n(\omega^\epsilon)}{n} \leq \liminf_{k \rightarrow \infty} \frac{\psi_{kn_1}(\omega^\epsilon)}{kn_1} \leq \Psi_{\min} + \epsilon.$$

□

6.2. Multifractal analysis of $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$.

Through the bi-Lipschitz homeomorphism $\pi_{\vec{w}}$, we have the following:

Theorem 13. *Let $\mathcal{B} = [\beta_*, \beta^*]$ and τ be defined as above. Define $\Lambda_\beta := \{x \in \Sigma_{\vec{w}} : d_{\mathcal{N}_{\alpha,V}}(x) = \beta\}$, then $\Lambda_\beta \neq \emptyset$ if and only if $\beta \in \mathcal{B}$. For any $\beta \in \mathcal{B}$*

$$\dim_H \Lambda_\beta = \tau^*(\beta) := \inf_{q \in \mathbb{R}} (\tau(q) + \beta q).$$

Moreover β_* is the optimal Hölder exponent of $\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}$ and

$$(6.10) \quad \beta_* = \gamma_{\mathcal{N}_{\alpha,V}|_{\Sigma_{\vec{w}}}} = \frac{\ln \alpha_\kappa}{-\Psi_{\min}}.$$

Proof. Given $\omega \in \Omega^{(\kappa)}$, write $x = \pi_{\vec{w}}(\omega)$. Since $\pi_{\vec{w}}$ is a bi-Lipschitz homeomorphism, we have $\underline{d}_{\mu_Q}(\omega) = \underline{d}_{\mathcal{N}_{\alpha,V}}(x)$ and $\bar{d}_{\mu_Q}(\omega) = \bar{d}_{\mathcal{N}_{\alpha,V}}(x)$. Then Theorem 13 follows from Theorem 12 and Proposition 4. \square

7. GLOBAL PICTURE

In this section we obtain the global picture and prove Theorem 1 (i), (ii) and (iii) by comparing any two different dynamical subsets $\Sigma_{\vec{v}}$ and $\Sigma_{\vec{w}}$.

7.1. Comparison of dynamical subsets.

Recall that for any $\alpha \in \mathcal{F}_\kappa$, we defined $\mathcal{D}(\alpha)$ according to (4.1). For any $\vec{w} \in \mathcal{D}(\alpha)$ fixed, we defined the potential $\Psi = \Psi_{\vec{w}}$. Let $\tilde{s}_{V,\vec{w}}$ be the root of $P(s\Psi_{\vec{w}}) = 0$. Let $d_{\vec{w}} = d_{\Psi_{\vec{w}}}$ be the weak Gibbs metric on $\Omega^{(\kappa)}$. Let $m_{\vec{w}}$ be the Gibbs measure with potential $\tilde{s}_{V,\vec{w}}\Psi_{\vec{w}}$. Let $\tilde{d}_{V,\vec{w}} = \dim_H^{\vec{w}} \mu_Q$ be the Hausdorff dimension of μ_Q on the metric space $(\Omega^{(\kappa)}, d_{\vec{w}})$. Let $\tilde{\gamma}_{V,\vec{w}}$ be the optimal Hölder exponent of μ_Q on the metric space $(\Omega^{(\kappa)}, d_{\vec{w}})$.

Now we fix $\alpha, \tilde{\alpha} \in \mathcal{F}_\kappa$. Define $\mathcal{D}(\alpha)$ and $\mathcal{D}(\tilde{\alpha})$ according to (4.1). Choose $\vec{w} \in \mathcal{D}(\alpha)$ and $\vec{v} \in \mathcal{D}(\tilde{\alpha})$, we will compare all the quantities related to dynamical subsets $\Sigma_{\vec{w}}$ and $\Sigma_{\vec{v}}$.

Theorem 14. *Fix $\alpha, \tilde{\alpha} \in \mathcal{F}_\kappa$ and $\vec{w} \in \mathcal{D}(\alpha)$, $\vec{v} \in \mathcal{D}(\tilde{\alpha})$. Then $d_{\vec{v}}$ is equivalent to $d_{\vec{w}}$, $m_{\vec{v}} = m_{\vec{w}}$ and*

$$\tilde{s}_{V,\vec{v}} = \tilde{s}_{V,\vec{w}}, \quad \tilde{d}_{V,\vec{v}} = \tilde{d}_{V,\vec{w}} \quad \text{and} \quad \tilde{\gamma}_{V,\vec{v}} = \tilde{\gamma}_{V,\vec{w}}.$$

Proof. At first we show that $d_{\vec{v}}$ is equivalent to $d_{\vec{w}}$. Given $\omega, \tilde{\omega} \in \Omega^{(\kappa)}$ and $\omega \neq \tilde{\omega}$. If $\omega_0 \neq \tilde{\omega}_0$, then

$$(7.1) \quad d_{\vec{v}}(\omega, \tilde{\omega}) = \text{diam}(\Sigma_{\tilde{\alpha},V}) \quad \text{and} \quad d_{\vec{w}}(\omega, \tilde{\omega}) = \text{diam}(\Sigma_{\alpha,V}).$$

Now assume $\omega_0 = \tilde{\omega}_0 = e$. Then

$$d_{\vec{v}}(\omega, \tilde{\omega}) = |B_{v^e \star \omega \wedge \tilde{\omega}}| \quad \text{and} \quad d_{\vec{w}}(\omega, \tilde{\omega}) = |B_{w^e \star \omega \wedge \tilde{\omega}}|.$$

Notice that $\#\mathcal{D}(\alpha)(\#\mathcal{D}(\tilde{\alpha}))$ is bounded by a constant only depending on $\alpha(\tilde{\alpha})$. It is also clear that the partial quotients of α and $\tilde{\alpha}$ are bounded by some constant only depending

on α and α , since $\alpha, \tilde{\alpha} \in \mathcal{F}_\kappa$. Thus by Theorem 3 we have

$$\frac{|B_{v^e \star \omega \wedge \tilde{\omega}}|}{|B_{w^e \star \omega \wedge \tilde{\omega}}|} \sim_{(\alpha, \tilde{\alpha}, V)} \frac{|B_{v^e}|}{|B_{w^e}|} \sim_{(\alpha, \tilde{\alpha}, V)} 1.$$

Together with (7.1) we conclude that $d_{\vec{v}}(\omega, \tilde{\omega})/d_{\vec{w}}(\omega, \tilde{\omega}) \sim 1$. That is, $d_{\vec{v}}$ and $d_{\vec{w}}$ are equivalent.

As a consequence the Hausdorff dimensions and the optimal Hölder exponents of μ_Q on $(\Omega, d_{\vec{v}})$ and $(\Omega, d_{\vec{w}})$ are equal. That is, $\tilde{d}_{V, \vec{v}} = \tilde{d}_{V, \vec{w}}$ and $\tilde{\gamma}_{V, \vec{v}} = \tilde{\gamma}_{V, \vec{w}}$.

Now we show that $\tilde{s}_{V, \vec{v}} = \tilde{s}_{V, \vec{w}}$ and $m_{\vec{v}} = m_{\vec{w}}$. Fix $\omega \in \Omega^{(\kappa)}$, By (4.3) we have

$$\psi_n^{(\vec{v})}(\omega) = \ln |B_{v^{\omega_0} \omega [1, \dots, n]}| \quad \text{and} \quad \psi_n^{(\vec{w})}(\omega) = \ln |B_{w^{\omega_0} \omega [1, \dots, n]}|.$$

By Theorem 3 we have

$$\frac{|B_{v^{\omega_0} \omega [1, \dots, n]}|}{|B_{w^{\omega_0} \omega [1, \dots, n]}|} \sim_{(\alpha, \tilde{\alpha}, V)} \frac{|B_{v^{\omega_0}}|}{|B_{w^{\omega_0}}|} \sim_{(\alpha, \tilde{\alpha}, V)} 1.$$

From this we conclude that

$$|\psi_n^{(\vec{v})}(\omega) - \psi_n^{(\vec{w})}(\omega)| \lesssim_{(\alpha, \tilde{\alpha}, V)} 1.$$

Now by Theorem 5 (ii), we conclude that $P(s\Psi_{\vec{v}}) = P(s\Psi_{\vec{w}})$ for any $s \in \mathbb{R}$, consequently they have the same zeros. That is, $\tilde{s}_{V, \vec{v}} = \tilde{s}_{V, \vec{w}} =: \tilde{s}_V$. Since we also have

$$|\tilde{s}_V \psi_n^{(\vec{v})}(\omega) - \tilde{s}_V \psi_n^{(\vec{w})}(\omega)| \lesssim_{(\alpha, \tilde{\alpha}, V)} 1,$$

still by Theorem 5 (ii), we have $m_{\vec{v}} = m_{\vec{w}}$. □

7.2. Proof of Theorem 1 (i), (ii) and (iii).

(i) Fix $\alpha_* \in \mathcal{F}_\kappa$ and $\vec{w}_* \in \mathcal{D}(\alpha_*)$, define

$$s_V(\kappa) := \tilde{s}_{V, \vec{w}_*}, \quad d_V(\kappa) := \tilde{d}_{V, \vec{w}_*} \quad \text{and} \quad \gamma_V(\kappa) := \tilde{\gamma}_{V, \vec{w}_*}.$$

Take any $\alpha \in \mathcal{F}_\kappa$, recall that $\Sigma_{\alpha, V} = \bigcup_{\vec{w} \in \mathcal{D}(\alpha)} \Sigma_{\vec{w}}$. By Theorem 14 and Theorem 9, for any $\vec{v} \in \mathcal{D}(\alpha)$ we have

$$\dim_H \Sigma_{\vec{v}} = \tilde{s}_{V, \vec{v}} = \tilde{s}_{V, \vec{w}_*} = s_V(\kappa).$$

Consequently

$$s_V(\alpha) = \dim_H \Sigma_{\alpha, V} = \dim_H \bigcup_{\vec{v} \in \mathcal{D}(\alpha)} \Sigma_{\vec{v}} = s_V(\kappa).$$

By Theorem 14, Theorem 10 and Theorem 11, for any $\vec{v} \in \mathcal{D}(\alpha)$ we have

$$\dim_H \mathcal{N}_{\alpha, V}|_{\Sigma_{\vec{v}}} = \tilde{d}_{V, \vec{v}} = \tilde{d}_{V, \vec{w}_*} = d_V(\kappa)$$

and $\mathcal{N}_{\alpha, V}|_{\Sigma_{\vec{v}}}$ is exact dimensional. Then by (1.7),

$$d_V(\alpha) = \dim_H \mathcal{N}_{\alpha, V} = d_V(\kappa).$$

By Theorem 14, Proposition 4 and Theorem 13, for any $\vec{v} \in \mathcal{D}(\alpha)$ we have

$$\gamma_{\mathcal{N}_{\alpha, V}|_{\Sigma_{\vec{v}}}} = \tilde{\gamma}_{V, \vec{v}} = \tilde{\gamma}_{V, \vec{w}_*} = \gamma_V(\kappa).$$

Then by (1.6),

$$\gamma_V(\alpha) = \gamma_{\mathcal{N}_{\alpha, V}} = \gamma_V(\kappa).$$

Then (1.8) holds.

(ii) The result follows from Definition 3, Theorem 9 and the fact that $\Sigma_{\alpha,V} = \bigcup_{\vec{w} \in \mathcal{D}(\alpha)} \Sigma_{\vec{w}}$.

(iii) The result follows from Definition 3, Proposition 3, Theorem 11 and the fact that $\Sigma_{\alpha,V} = \bigcup_{\vec{w} \in \mathcal{D}(\alpha)} \Sigma_{\vec{w}}$. \square

Remark 5. (1.8) has the following advantage: to compute these three quantities, we can choose special element in \mathcal{F}_κ to make the computation easier. Indeed in next section, we will always pick $\alpha_\kappa \in \mathcal{F}_\kappa$ to do the computation. Moreover, due to Theorem 14, we can fix any $w \in \mathcal{D}(\alpha_\kappa)$, and compute $\gamma_V(\kappa)$, $d_V(\kappa)$, $s_V(\kappa)$ by (6.10), (5.8) and (4.7), respectively.

8. ASYMPTOTIC PROPERTIES AND THE CONSEQUENCES

In this section we discuss the asymptotic properties of $\gamma_V(\kappa)$, $s_V(\kappa)$ and $d_V(\kappa)$ when $V \rightarrow \infty$. In particular, we finish the proof of Theorem 1 (iv) and (v).

By Remark 5, in this section we always fix $\alpha = \alpha_\kappa$ and some $w \in \mathcal{D}(\alpha_\kappa)$. Recall that we simplify $\mathcal{A}_\kappa = \{e_{\kappa,1}, \dots, e_{\kappa,2\kappa+2}\}$ to $\{e_1, \dots, e_{2\kappa+2}\}$.

8.1. Asymptotic property of $\gamma_V(\kappa)$.

At first we note that Lemma 5 implies the following useful fact: There exists a constant $c = c_\kappa > 1$ such that for any $w \in \Omega_n^{(\alpha_\kappa)}$,

$$(8.1) \quad c^{-n} V^{-(\kappa-2)|w|_{e_{\kappa+2}}-n} \leq |B_w| \leq c^n V^{-(\kappa-2)|w|_{e_{\kappa+2}}-n},$$

where $|w|_{e_{\kappa+2}}$ stands for $\#\{1 \leq i \leq n : w_i = e_{\kappa+2}\}$. The proof is a direct computation by noticing that $V > 20$.

Proposition 5.

$$\lim_{V \rightarrow \infty} \gamma_V(1) \ln V = \frac{3}{2} \ln \alpha_1 =: \hat{\rho}_1 \quad \text{and} \quad \lim_{V \rightarrow \infty} \gamma_V(\kappa) \ln V = \frac{2}{\kappa} \ln \alpha_\kappa =: \hat{\rho}_\kappa \quad (\kappa \geq 2).$$

Proof. Fix some $\vec{w} \in \mathcal{D}(\alpha_\kappa)$ and define Ψ according to (4.3). By Remark 5,

$$(8.2) \quad \gamma_V(\kappa) = \frac{\ln \alpha_\kappa}{-\Psi_{\min}}.$$

Thus we only need to estimate Ψ_{\min} . Recall that $\psi_n(\omega) = \ln |B_{w^{\omega_0 \omega} [1, \dots, n]}| = \ln |B_{w^{\omega_0 \star \omega} |_n}|$ and $\psi_{n,\min} = \min\{\psi_n(\omega) : \omega \in \Omega^{(\kappa)}\}$. Thus $\exp(\psi_{n,\min})$ is just the minimal length of the bands $\{B_u : |u| = n + N_\kappa + 1, w^{e_j} \prec u \text{ for some } j\}$.

At first we assume $\kappa = 1$. Assume $u = w^{e_j} v$ with $|v| = n$. Then by (8.1)

$$(8.3) \quad c^{-n} V^{|v|_{e_3}-n} \lesssim |B_u| \lesssim c^n V^{|v|_{e_3}-n}$$

Notice that $e_1 e_3 e_4 e_1$ is admissible. Take $\tilde{u} = w^{e_4} \tilde{v}$ such that $|\tilde{v}| = n$ and $\tilde{v} \prec (e_1 e_3 e_4)^\infty$. Then $|\tilde{v}|_{e_3} \leq n/3 + 1$. Then (8.3) implies that there exists some constant C (depending on κ, N_κ, V) such that

$$\psi_{n,\min} \leq \ln |B_{\tilde{u}}| \leq C + n \ln c - \frac{2n}{3} \ln V.$$

On the other hand for any $u = w^{e_j}v$ with $|v| = n$, by the definition of the incidence matrix A_1 , it is ready to show that $|v|_{e_3} \geq n/3 - 1$. Then (8.3) implies that there exists some constant C' (depending on κ, N_κ, V) such that

$$\psi_{n,\min} \geq \min_u \ln |B_u| \geq C' - n \ln c - \frac{2n}{3} \ln V.$$

Consequently

$$-\ln c - \frac{2}{3} \ln V \leq \Psi_{\min} \leq \ln c - \frac{2}{3} \ln V.$$

Now by (8.2) we conclude that $\hat{\varrho}_1 = \frac{3}{2} \ln \alpha_1$.

Next we assume $\kappa \geq 2$. Assume $u = w^{e_j}v$ with $|v| = n$. Then by (8.1)

$$(8.4) \quad c^{-n} V^{-(\kappa-2)|v|_{e_{\kappa+2}}-n} \lesssim |B_u| \lesssim c^n V^{-(\kappa-2)|v|_{e_{\kappa+2}}-n}.$$

Notice that $\kappa - 2 \geq 0$ and $e_1 e_{\kappa+2} e_1$ is admissible. Take $\tilde{u} = w^{e_{\kappa+2}} \tilde{v}$ such that $|\tilde{v}| = n$ and $\tilde{v} \prec (e_1 e_{\kappa+2})^\infty$. Then $|\tilde{v}|_{e_{\kappa+2}} \geq n/2 - 1$. Then (8.4) implies that there exists some constant \tilde{C} such that

$$\psi_{n,\min} \leq \ln |B_{\tilde{u}}| \leq \tilde{C} + n \ln c - \frac{\kappa n}{2} \ln V.$$

On the other hand for any $u = w^{e_j}v$ with $|v| = n$, since $e_{\kappa+2} e_{\kappa+2}$ is not admissible, we have $|v|_{e_{\kappa+2}} \leq n/2 + 1$. Then (8.4) implies that there exists some constant \tilde{C}' such that

$$\psi_{n,\min} \geq \min_u \ln |B_u| \geq \tilde{C}' - n \ln c - \frac{\kappa n}{2} \ln V.$$

Consequently

$$-\ln c - \frac{\kappa}{2} \ln V \leq \Psi_{\min} \leq \ln c - \frac{\kappa}{2} \ln V.$$

Now by (8.2) we conclude that $\hat{\varrho}_\kappa = \frac{2}{\kappa} \ln \alpha_\kappa$. □

Remark 6. When $\kappa = 1, 2$, we have

$$\hat{\varrho}_1 = \frac{3}{2} \ln \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \hat{\varrho}_2 = \ln(\sqrt{2} + 1).$$

8.2. Asymptotic property of $d_V(\kappa)$.

Proposition 6.

$$\lim_{V \rightarrow \infty} d_V(\kappa) \ln V = \frac{\kappa \alpha_\kappa + 2}{2\alpha_\kappa(\alpha_\kappa - 1)} \ln \alpha_\kappa =: \varrho_\kappa.$$

Proof. By (5.8) we have

$$d_V(\kappa) = \dim_H \mu_Q = \frac{\ln \alpha_\kappa}{-\Psi_*(\mu_Q)}.$$

Now we study $\Psi_*(\mu_Q)$. Since μ_Q is ergodic, by Kingman's sub-additive ergodic theorem, for μ_Q a.e. $\omega \in \Omega^{(\kappa)}$, we have

$$-\frac{\psi_n(\omega)}{n} \rightarrow -\Psi_*(\mu_Q).$$

Recall that by the definition (4.3), $\psi_n(\omega) = \ln |B_{w^{\omega_0 \omega} [1, \dots, n]}|$. By (8.1) we have

$$(8.5) \quad c^{-n} V^{-(\kappa-2)|\omega|_n|_{e_{\kappa+2}}-n} \lesssim |B_{w^{\omega_0 \omega} [1, \dots, n]}| \lesssim c^n V^{-(\kappa-2)|\omega|_n|_{e_{\kappa+2}}-n}.$$

To get the value $-\Psi_*(\mu_Q)$, we need to know the frequency of $e_{\kappa+2}$ in a μ_Q typical point ω . Define $\varphi(\omega) = \chi_{[e_{\kappa+2}]}(\omega)$. Since μ_Q is ergodic, by (5.4) for μ_Q a.e. $\omega \in \Omega^{(\kappa)}$ we have

$$\frac{\#\{1 \leq j \leq n : \omega_j = e_{\kappa+2}\}}{n} = \frac{S_n \varphi(\omega)}{n} \rightarrow \int_{\Omega} \varphi d\mu_Q = \mu_Q([e_{\kappa+2}]) = p_{e_{\kappa+2}} = \frac{\alpha_{\kappa}}{\kappa\alpha_{\kappa} + 2}.$$

Combining with (8.5) we conclude that

$$\frac{2\alpha_{\kappa}(\alpha_{\kappa} - 1)}{\kappa\alpha_{\kappa} + 2} \ln V - \ln c \leq -\Psi_*(\mu_Q) \leq \frac{2\alpha_{\kappa}(\alpha_{\kappa} - 1)}{\kappa\alpha_{\kappa} + 2} \ln V + \ln c.$$

Now combining with the dimension formula we get the result. \square

Remark 7. When $\kappa = 1, 2$, we have

$$\varrho_1 = \frac{5 + \sqrt{5}}{4} \ln \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \varrho_2 = \ln(\sqrt{2} + 1).$$

8.3. Asymptotic property of $s_V(\kappa)$.

The asymptotic properties of $s_V(\kappa)$ has been studied in [27, 19, 28]. Let us recall the result.

For any $0 \leq x \leq 1$ define

$$\mathbf{R}(x) := \begin{pmatrix} 0 & x^{(\kappa-1)} & 0 \\ (\kappa+1)x & 0 & \kappa x \\ \kappa x & 0 & (\kappa-1)x \end{pmatrix}$$

Let $\psi(x)$ be the spectral radius of $\mathbf{R}(x)$. Then it is seen that $\psi(0) = 0$, $\psi(1) = \alpha_{\kappa}$ and $\psi(x)$ is continuous and strictly increasing. Assume x_{κ} is the unique number such that $\psi(x_{\kappa}) = 1$, then

Theorem 15 ([27, 19, 28]).

$$\lim_{V \rightarrow \infty} s_V(\kappa) \ln V = -\ln x_{\kappa} =: \rho_{\kappa}.$$

In the following we will make x_{κ} explicit. It is seen that for any $x > 0$, the matrix $\mathbf{R}(x)$ is primitive, thus the spectral radius $\psi(x)$ of $\mathbf{R}(x)$ is the largest positive eigenvalue of $\mathbf{R}(x)$. By a direct computation, we have

$$\det(\lambda I_3 - \mathbf{R}(x)) = \lambda^3 - (\kappa - 1)x\lambda^2 - (\kappa + 1)x^{\kappa}\lambda - x^{\kappa+1}.$$

Thus x_{κ} is the unique number in $(0, 1)$ such that

$$1 - (\kappa - 1)x_{\kappa} - (\kappa + 1)x_{\kappa}^{\kappa} - x_{\kappa}^{\kappa+1} = 0.$$

Write $y_{\kappa} = 1/x_{\kappa}$, then y_{κ} is the unique number in $(1, \infty)$ such that

$$y_{\kappa}^{\kappa+1} - (\kappa - 1)y_{\kappa}^{\kappa} - (\kappa + 1)y_{\kappa} - 1 = 0.$$

We claim that $\kappa - 1 < y_{\kappa} < \kappa$ when $\kappa \geq 3$. Indeed define

$$F(y) := y^{\kappa+1} - (\kappa - 1)y^{\kappa} - (\kappa + 1)y - 1,$$

then $F(\kappa - 1) = -\kappa^2 < 0$ and $F(\kappa) = \kappa^{\kappa} - \kappa(\kappa + 1) - 1 > 0$. Thus $F(y) = 0$ has a root in $(\kappa - 1, \kappa)$. On the other hand we know that $F(y) = 0$ has only one root y_{κ} in $(1, \infty)$, thus $\kappa - 1 < y_{\kappa} < \kappa$.

Remark 8. When $\kappa = 1, 2$, by direct computation we have

$$\rho_1 = \rho_2 = \ln(\sqrt{2} + 1).$$

8.4. Proof of Theorem 1 (iv) and (v).

At first we show (iv). The three asymptotic properties have been established by Proposition 5, Proposition 6 and Theorem 15. By Remark 6, 7 and 8 we have $\hat{\varrho}_2 = \varrho_2 = \rho_2 = \ln(1 + \sqrt{2})$. When $\kappa \neq 2$, it is direct to verify that $\hat{\varrho}_\kappa < \varrho_\kappa$.

Now we show that $\varrho_\kappa < \rho_\kappa$ for any $\kappa \neq 2$.

At first we claim that $\kappa < \alpha_\kappa < \kappa + 1$. Indeed define $G(x) = x^2 - \kappa x - 1$, then $G(\kappa) = -1$ and $G(\kappa + 1) = \kappa > 0$, thus $G(x) = 0$ has a root in $(\kappa, \kappa + 1)$. On the other hand $G(x)$ has a unique positive root, which is α_κ , thus we conclude that $\kappa < \alpha_\kappa < \kappa + 1$.

Write $\delta_\kappa := \frac{\kappa\alpha_\kappa + 2}{2\alpha_\kappa(\alpha_\kappa - 1)}$. We claim that $\delta_\kappa \leq 2/3$ when $\kappa \geq 8$. Indeed for $\kappa \geq 2$ we have

$$\delta_\kappa = \frac{\kappa\alpha_\kappa + 2}{2\alpha_\kappa(\alpha_\kappa - 1)} \leq \frac{\kappa(\kappa + 1) + 2}{2\kappa(\kappa - 1)}.$$

By a simple computation we get $\delta_\kappa \leq 2/3$ for $\kappa \geq 8$. As a result for $\kappa \geq 8$ we have

$$e^{\varrho_\kappa} = \alpha_\kappa^{\delta_\kappa} \leq (\kappa + 1)^{2/3}.$$

On the other hand

$$e^{\rho_\kappa} = y_\kappa > \kappa - 1.$$

Thus for $\kappa \geq 8$ we have

$$e^{\varrho_\kappa} \leq (\kappa + 1)^{2/3} < \kappa - 1 < e^{\rho_\kappa}.$$

That is, $\varrho_\kappa < \rho_\kappa$ for $\kappa \geq 8$.

By direct computation we get $\varrho_\kappa < \rho_\kappa$ for $1 \leq \kappa < 8$ and $\kappa \neq 2$. Thus (iv) follows.

Now we show (v). Assume $\kappa \neq 2$, then $\hat{\varrho}_\kappa < \varrho_\kappa < \rho_\kappa$. By the definition we have

$$\lim_{V \rightarrow \infty} \gamma_V(\kappa) \ln V < \lim_{V \rightarrow \infty} d_V(\kappa) \ln V < \lim_{V \rightarrow \infty} s_V(\kappa) \ln V.$$

Consequently there exists $V_0(\kappa) > 20$ such that for any $V \geq V_0(\kappa)$,

$$\gamma_V(\kappa) \ln V < d_V(\kappa) \ln V < s_V(\kappa) \ln V.$$

That is, $\gamma_V(\kappa) < d_V(\kappa) < s_V(\kappa)$ for $V \geq V_0(\kappa)$. □

9. APPENDIX

In this appendix, we give another proof of the fact that $d_V(\kappa) < s_V(\kappa)$ for $V \geq V_0(\kappa)$ when $\kappa \neq 2$. This proof is more elementary and has the advantage that the constant $V_0(\kappa)$ can be estimated explicitly.

By (4.7) and (5.6) we have

$$d_V(\kappa) = \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_*(\mu_Q)} \quad \text{and} \quad s_V(\kappa) = \dim_H m = \frac{h_m(\sigma)}{-\Psi_*(m)}.$$

At first we claim that if $\mu_Q \neq m$, then $d_V(\kappa) < s_V(\kappa)$. Indeed by Theorem 5 (iii), the unique equilibrium state of $s_V(\kappa)\Psi$ is m . Since $\mu_Q \neq m$ we have

$$h_{\mu_Q}(\sigma) + s_V(\kappa)\Psi_*(\mu_Q) < h_m(\sigma) + s_V(\kappa)\Psi_*(m) = P(s_V(\kappa)\Psi) = 0.$$

Since $\Psi_*(\mu_Q) < 0$ we conclude that

$$d_V(\kappa) = \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_*(\mu_Q)} < s_V(\kappa).$$

Thus we only need to study when $\mu_Q \neq m$. For this purpose we consider two words $u^n = ((II, 1)_\kappa(I, 1)_\kappa)^{3n} = (e_{\kappa+2}e_1)^{3n}$ and $\tilde{u}^n = ((II, 1)_\kappa(III, 1)_\kappa(I, 1)_\kappa)^{2n} = (e_{\kappa+2}e_{\kappa+3}e_1)^{2n}$. We will estimate respectively the following

$$\mu_Q([e_1u^n]), \quad \mu_Q([e_1\tilde{u}^n]), \quad m([e_1u^n]) \quad \text{and} \quad m([e_1\tilde{u}^n]).$$

At first by (5.7) we have

$$\mu_Q([e_1u^n]) \sim \alpha_\kappa^{-6n-1} \quad \text{and} \quad \mu_Q([e_1\tilde{u}^n]) \sim \alpha_\kappa^{-6n-1}.$$

Consequently

$$(9.1) \quad \frac{\mu_Q([e_1\tilde{u}^n])}{\mu_Q([e_1u^n])} \sim 1.$$

Next we estimate $m([e_1u^n])$ and $m([e_1\tilde{u}^n])$. Since m is the Gibbs measure with potential $s_V(\kappa)\Psi$, we have

$$m([e_1u^n]) \sim |B_{w^{e_1}u^n}|^{s_V(\kappa)} \quad \text{and} \quad m([e_1\tilde{u}^n]) \sim |B_{w^{e_1}\tilde{u}^n}|^{s_V(\kappa)}.$$

Now we estimate $|B_{w^{e_1}u^n}|$ and $|B_{w^{e_1}\tilde{u}^n}|$. At first we have

$$3n \leq |w^{e_1}u^n|_{e_{\kappa+2}} \leq N + 3n \quad \text{and} \quad 2n \leq |w^{e_1}\tilde{u}^n|_{e_{\kappa+2}} \leq N + 2n.$$

By (8.1) we have

$$\begin{cases} c^{-6n}V^{-3\kappa n} & \lesssim |B_{w^{e_1}u^n}| \lesssim c^{6n}V^{-3\kappa n} \\ c^{-6n}V^{-2(\kappa+1)n} & \lesssim |B_{w^{e_1}\tilde{u}^n}| \lesssim c^{6n}V^{-2(\kappa+1)n} \end{cases}$$

As a consequence we get

$$(9.2) \quad C_{V,\kappa}^n := (c^{-12}V^{2-\kappa})^n \lesssim \frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}^n}|} \lesssim (c^{12}V^{2-\kappa})^n =: D_{V,\kappa}^n.$$

Note that $c = c_\kappa$ is a constant only depending on κ . Define $V_0(\kappa) := c_\kappa^{12}$. By (9.2), it is direct to check that for $\kappa = 1$, if $V > V_0(1)$, then $C_{V,1} > 1$; for $\kappa \geq 3$, if $V > V_0(\kappa)$, then $D_{V,\kappa} < 1$. Consequently if $V > V_0(\kappa)$, then

$$\begin{aligned} \frac{m([e_1u^n])}{m([e_1\tilde{u}^n])} &\sim \left(\frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}^n}|} \right)^{s_V(\kappa)} \gtrsim (C_{V,1})^{ns_V(\kappa)} \rightarrow \infty, \quad (n \rightarrow \infty) \quad \kappa = 1 \\ \frac{m([e_1u^n])}{m([e_1\tilde{u}^n])} &\sim \left(\frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}^n}|} \right)^{s_V(\kappa)} \lesssim (D_{V,\kappa})^{ns_V(\kappa)} \rightarrow 0, \quad (n \rightarrow \infty) \quad \kappa \geq 3. \end{aligned}$$

Combine with (9.1) we conclude that $\mu_Q \neq m$. Then the result follows. \square

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(Y.-H. QU) DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, BEIJING 100084, P. R. CHINA.

E-mail address: yhqu@math.tsinghua.edu.cn; yanhui.qu@gmail.com